Video: A Proof of the Walrasian Equilibrium Existence Theorem

I've recorded a series of videos in which I wrote out a proof of the Walrasian Equilibrium Existence Theorem and described what I was doing (and why) as I wrote. This pdf document contains the handwritten notes that I created in those videos.

The series of videos can be accessed from this web page:

markwalkereconomics.com/Video/2013ExistenceVideosPage.htm

I've numbered the videos, V1 - V8 (plus a correction, numbered V9). The notes include the same numbers: each number appears in the notes at the point where the corresponding video begins.

As it says on the web page, you can either stream the videos (in which case they should begin playing immediately), or you can download them (in which case a video won't begin playing until the download is complete). If you download a video, it will be stored on your computer. If for any reason streaming doesn't work, downloading should work fine.

If you have any problem viewing or downloading these videos, please let me know.

V1

EXISTENCE OF EQUILIBRIUM

DEFN: A WALRASIAN EQUILIBRIUM OF AN ECONOMY $E = ((u^i, \mathring{x}^i))_1^n$ IS A PAIR $(p^*, (x^{*i})_1^n) \in \mathbb{R}_+^{\ell} \times \mathbb{R}_+^{n\ell}$ THAT SATISFIES

(u-MAX) → $\forall i \in N: x^{*i} \in D^i(p^*) = \{x^i \in \mathbb{R}_+^{\ell} \mid x^i \text{ max's } u^i \text{ on } B(p^*, \mathring{x}^i)\}$

(M-CLR) → $\forall k = 1, \dots, \ell: \sum_{i=1}^{n} x_k^{*i} \leqq \sum_{i=1}^{n} \mathring{x}_k^i$ AND $p_k^* > 0 \Rightarrow \sum_{i=1}^{n} x_k^{*i} = \sum_{i=1}^{n} \mathring{x}_k^i$.

THEOREM: IF EACH CONSUMER $(u^i, \mathring{x}^i)$ SATISFIES

(a) u^i IS CONTINUOUS, LNS, QUASICONCAVE

AND (b) $\mathring{x}_k^i > 0, \quad k = 1, \dots, \ell,$

THEN THE ECONOMY $E = ((u^i, \mathring{x}^i))_1^n$ HAS A WALRASIAN EQUILIBRIUM (WE).

$\mathring{x}_k = \sum_{i=1}^{n} \mathring{x}^i_k$

V2 PROOF:

LET $\beta = 1 + \max\{\mathring{x}_1, \mathring{x}_2, \ldots, \mathring{x}_\ell\}$

AND $K = \{x \in \mathbb{R}^\ell_+ \mid -\beta \leqq x_k \leqq \beta \ (k=1,\ldots,\ell)\}$.

FOR EACH $i \in N$, DEFINE

SEE FIGURE 1

$$\hat{B}^i(p) = B(p, \mathring{x}^i) \cap K = \{x^i \in K \mid p \cdot x^i \leqq p \cdot \mathring{x}^i\}$$

$$\hat{D}^i(p) = \{x^i \in K \mid x^i \text{ MAX'S } u^i \text{ ON } \hat{B}^i(p)\}$$

$$\hat{z}^i(p) = \hat{D}^i(p) - \{\mathring{x}^i\}$$

DEFINE $\hat{z}(p) = \sum_{i=1}^{n} \hat{z}^i(p)$.

V3 WE WILL SHOW:

(1) THE MARKET EXCESS DEMAND CORRESPONDENCE $\hat{z}(\cdot) : S \twoheadrightarrow \mathbb{R}^\ell_+$ HAS AN EQUILIBRIUM:

$$\exists\, p^* \in S,\ z^* \in K : \begin{cases} z^* \in \hat{z}(p) \\ \sum_{i=1}^{n} z^{*i} \leqq 0 \\ p^*_k > 0 \Rightarrow z^*_k = 0 \end{cases}$$

$z^* \leqq 0$ $[z^*_k \leqq 0, \forall k]$

(2) p^* AND z^* GIVE US AN ALLOCATION $(x^{*i})_1^n$ SUCH THAT $(p^*, (x^{*i})_1^n)$ IS A WE OF $\hat{E}$.

(3) $(p^*, (x^{*i})_1^n)$ IS A WE OF THE ACTUAL ECONOMY E.

V4 (1) $\hat{z}(\cdot): S \twoheadrightarrow K$ HAS A MARKET EQUILIBRIUM:

DEFINE A "PRICE ADJUSTMENT CORRESPONDENCE"

$$\mu: K \twoheadrightarrow S$$

AS FOLLOWS:

$$\forall z \in K: \quad \mu(z) = \{ p \in S \mid p \text{ max's } p \cdot z \text{ on } S \}$$

PUTS ALL THE WEIGHT OF PRICES ON THE GOOD(S) WITH THE LARGEST z_h-VALUE;

e.g., IF $z = (3, -1, 3, 2, 0)$,

THEN $\mu(z)$ HAS p's s.t. $p_1 + p_3 = 1$

AND $p_2 = p_4 = p_5 = 0$

DEFINE A "STATE TRANSITION CORRESPONDENCE"

$f: S \times K \twoheadrightarrow S \times K$ AS FOLLOWS:

$$f(p, z) = \mu(z) \times \hat{z}(p)$$

$$= \{ (p, z) \in S \times K \mid p \in \mu(z) \text{ AND } z \in \hat{z}(p) \}.$$

WE WILL SHOW THAT

(1.1) f HAS A FIXED POINT (p^*, z^*)

(1.2) A FIXED POINT (p^*, z^*) OF f IS AN EQUILIBRIUM OF $\hat{z}(\cdot)$.

V5 (1.1) $f: S \times K \twoheadrightarrow S \times K$ HAS A FIXED POINT:

(1.1.a) APPLY THE MAXIMUM THEOREM TO μ:

$x \in X$ IS $p \in S$ } $S, K \neq \emptyset$, COMPACT
$e \in E$ IS $z \in K$ }

u IS $p \cdot z$ ← CONT'S

$\varphi(e)$ IS S ← CONSTANT, ∴ CONT'S

$\mu(e)$ IS $\mu(z)$

∴ μ IS UHC (HAS A CLOSED GRAPH)

(1.1.b) APPLY THE MAXIMUM THEOREM TO $\hat{z}^i$ ($\forall i$):

$x \in X$ IS $x^i \in K$ } $S; K \neq \emptyset$, COMPACT
$e \in E$ IS $p \in S$ }

u IS $u^i(\cdot, p)$ ← CONT'S

$\varphi(e)$ IS $\hat{B}^i(p)$ ← $\neq \emptyset$, $\hat{B}^i$ IS A CONT'S CORRESPONDENCE ← EXERCISE

$\mu(e)$ IS $\hat{D}^i(p)$

∴ ~~IS UHC~~ (CLOSED GRAPH) ← $\hat{D}^i$ IS UHC

∴ $\hat{z}^i(p) = \hat{D}^i(p) - \{\mathring{x}^i\}$ IS UHC, $\forall i$

∴ $\hat{z} = \sum_{i=1}^{n} \hat{z}^i$ IS UHC.

(1.1.c) APPLY KAKUTANI'S THEOREM TO $f: S \times K \twoheadrightarrow S \times K$

$S \times K \neq \emptyset$, COMPACT, CONVEX

$f = \mu \times \hat{z}$, UHC (CLOSED GRAPH)

$\forall (p, z) \in S \times K$: $f(p,z) = \mu(z) \times \hat{z}(p)$ IS $\neq \emptyset$ AND IS CONVEX SET. ← EXERCISE

∴ f HAS A FIXED POINT (p^*, z^*).

i.e., $z_k^* \leqq 0, \forall k$

V6 (1.2) WE SHOW THAT $z^* \leqq \underline{0}$ AND $p_k^* > 0 \Rightarrow z_k^* = 0$:

SINCE $p^* \in \mu(z^*)$, p^* MAX'S $p \cdot z^*$ ON S

SINCE $p^* \in S$, WE HAVE $p_m^* > 0$ FOR SOME m

AND $p^* \in \mu(z^*) \Rightarrow z_m^* = \max\{z_1^*, \ldots, z_\ell^*\}$.

SUPPOSE $z_m^* > 0$; THEN $p_m^* z_m^* > 0$.

BUT ALSO $z_k^* < z_m^* \Rightarrow p_k^* = 0$, $\forall k$

$\therefore$ $p_k^* z_k^* \geqq 0$, $k = 1, \ldots, \ell$.

WL ENSURES THAT $p^* \cdot z^* = 0$ $(p_1^* z_1^* + \cdots + p_\ell^* z_\ell^* = 0)$.

$\therefore$ $p_k^* z_k^* = 0$, $\forall k$.

$\therefore$ $p_m^* z_m^* = 0$, $\therefore$ $z_m^* = 0$, $\therefore$ $z_k^* \leqq 0$, $\forall k$

$p_k^* > 0 \Rightarrow z_k^* = 0$.

V7 (2) WE DISAGGREGATE THE NET DEMAND BUNDLE z^* INTO AN ALLOCATION $(x^{*i})_1^n = (x^{*1}, \ldots, x^{*n})$ s.t. $(p^*, (x^{*i})_1^n)$ IS A WE OF $\hat{E}$:

SINCE $z^* \in \hat{z}(p^*) = \sum_{i=1}^n \hat{z}^i(p^*)$, THEN BY DEFN THERE ARE $z^{*1}, \ldots, z^{*n}$ s.t.

(2.1) $\boxed{z^{*i} \in \hat{z}^i(p^*),\ \forall i}$ AND (2.2) $\boxed{\sum_{i=1}^n z^{*i} = z^*.}$

LET $x^{*i} = \overset{\circ}{x}{}^i + z^{*i}$ (i.e., $z^{*i} = x^{*i} - \overset{\circ}{x}{}^i$), $\forall i$.

FOR $\hat{E}$ THEN

(u-MAX) (2.1) $\forall i$: $x^{*i} \in \hat{D}^i(p^*)$ — i.e., x^{*i} MAX's u^i ON $\hat{B}^i(p^*)$

(m-CLR) (2.2) $\forall k$: $\sum_{i=1}^n x_k^{*i} \leq \sum_{i=1}^n \overset{\circ}{x}{}_k^i$ AND $p_k^* > 0 \Rightarrow \sum_{i=1}^n x_k^{*i} = \sum_{i=1}^n \overset{\circ}{x}{}_k^i$.

ALSO (m-CLR) FOR E

(3) WE SHOW THAT $\forall i$: $x^{*i} \in \overset{\circ}{D}{}^i(p^*)$, NOT JUST IN $\hat{D}^i(p^*)$:

WE FIRST SHOW THAT $x^{*i} \in \operatorname{int} K$:

WE HAVE

$$x^{*i}_k \leqq \Sigma_1^n x^{*i}_k \leqq \Sigma_1^n \overset{\circ}{x}{}^i_k = \overset{\circ}{x}_k < \beta$$

$\therefore x^{*i} \in \operatorname{int} K$.

NOW SUPPOSE THAT $x^{*i} \notin \overset{\circ}{D}{}^i(p^*)$.

SINCE $x^{*i} \notin \overset{\circ}{D}{}^i(p^*)$, $\exists \tilde{x}^i \in \mathbb{R}^\ell_+$ s.t.

$$p^* \cdot \tilde{x}^i \leqq p^* \cdot \overset{\circ}{x}{}^i \text{ AND } u^i(\tilde{x}^i) > u^i(x^{*i}).$$

SINCE $B(p^*, \overset{\circ}{x}{}^i)$ IS CONVEX, EVERY BUNDLE x^i IN THE LINE SEGMENT $[\tilde{x}^i, x^{*i}]$ IS IN $B(p^*, \overset{\circ}{x}{}^i)$, AND SINCE $u^i(\tilde{x}^i) > u^i(x^{*i})$ AND u^i IS CONTINUOUS AND ~~[illegible]~~, $x^i \in [\tilde{x}^i, x^{*i}] \Rightarrow u^i(x^i) > u^i(x^{*i})$.

EXERCISE

SEE FIGURES 2 AND 3

QUASI-CONCAVE

WE HAVE $x^i \in B(p^*, \overset{\circ}{x}{}^i) \cap K = \hat{B}^i(p^*)$ AND $u^i(x^i) > u^i(x^{*i})$ — x^i IS IN K, AFFORDABLE, AND STRICTLY BETTER THAN x^{*i};

$\therefore x^{*i} \notin \hat{D}^i(p^*)$, A CONTRADICTION, FROM (2).

$\therefore x^{*i} \in \overset{\circ}{D}{}^i(p^*)$, $\forall i$

$\forall i$: $x^{*i} \in \overset{\circ}{D}{}^i(p^*)$, FROM (3)

$\forall k$: $\Sigma_1^n x^{*i}_k \leqq \Sigma_1^n \overset{\circ}{x}{}^i_k$, AND $=$ IF $p^*_k > 0$, FROM (2).

$\therefore (p^*, (x^{*i})_1^n)$ IS A WE FOR $\mathcal{E}$. ||

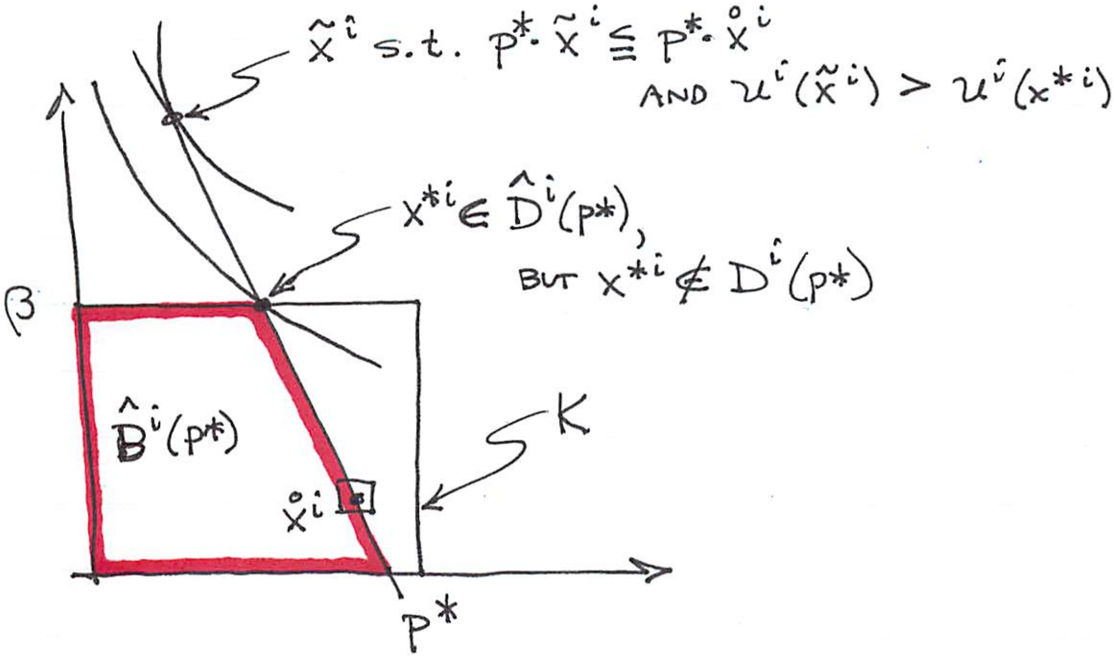
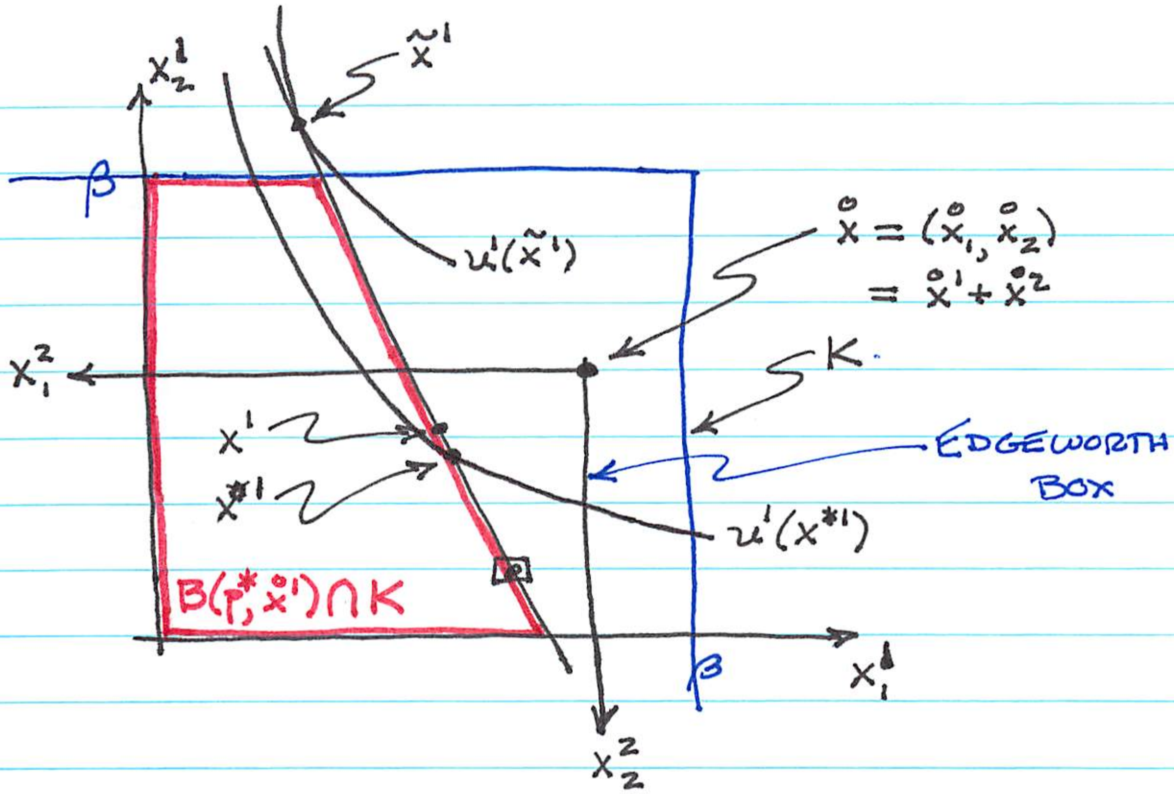


FIGURE 1:

$D^i(p^*) = \{\tilde{x}^i\}$, BUT $\hat{D}^i(p^*) = \{x^{*i}\}$



IF $x^{*1} \notin D^1(p^*)$, THEN $\exists\, \tilde{x}^1$: $\tilde{x}^1 \in B(p^*, \mathring{x}^1)$ AND $u^1(\tilde{x}^1) > u^1(x^{*1})$.

$x^1 \in [\tilde{x}^1, x^{*1}]$, $\therefore$ $x^1 \in B(p^*, \mathring{x}^1)$ AND $u^1(x^1) > u(x^{*1})$.

$x^1 \in \text{nbd}(x^{*1})$, $\therefore$ $x^1 \in K$.

$\therefore$ $x^1 \in B(p^*, \mathring{x}^1) \cap K$ AND $u^1(x^1) > u^1(x^{*1})$,

$\therefore$ $x^{*1} \notin \hat{D}^1(p^*)$, A CONTRADICTION.

FIGURE 2: THE ARGUMENT SHOWING THAT $x^{*i} \in D^i(p^*)$.

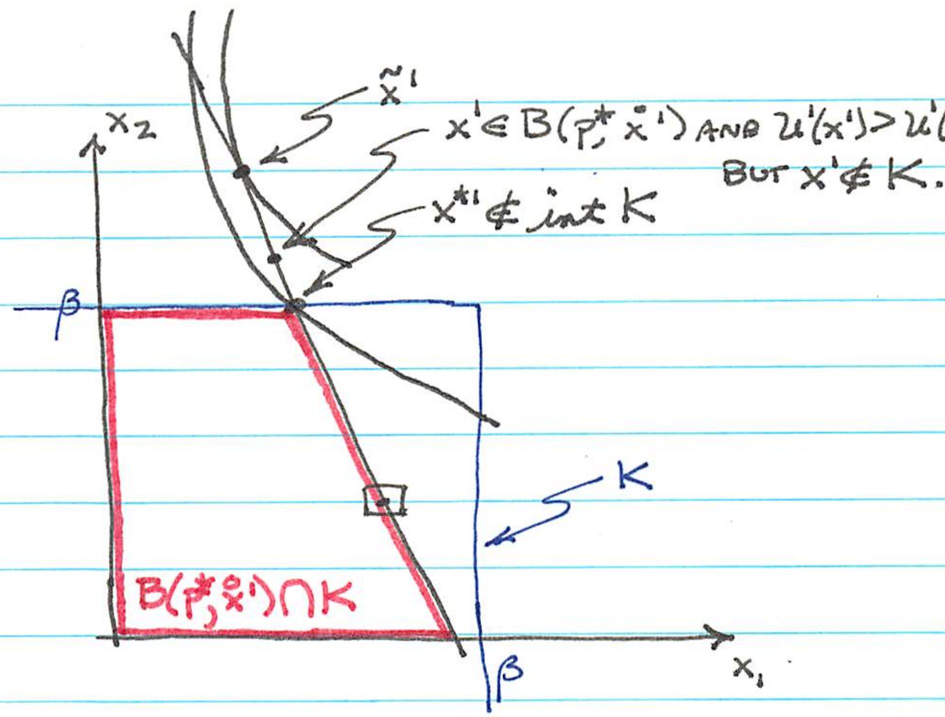


FIGURE 3: THE ARGUMENT DOESN'T WORK IF $x^{*i} \notin \text{int } K$.

(THIS IS WHY WE USED $1+\beta$ INSTEAD OF β IN DEFINING K.)