Lecture Notes

on

The Core

Economics 501B

University of Arizona

Fall 2014

The Walrasian Model's Assumptions

The following assumptions are implicit rather than explicit in the Walrasian model we've developed:

(1) Each individual unit (each household or firm) is a ***price-taker*** — *i.e.*, each one behaves as if its own actions will not affect prices. We might expect this to hold, for example, if there are many buyers and sellers in each market. How many? Some empirical evidence suggests that the number doesn't have to be very large, perhaps as few as five or ten on each side of the market in many cases. Note, though, that this will also depend on what is "the market." Strictly speaking, each market would have to be for a single, homogeneous good — a market in which consumers are indifferent between goods sold by any two sellers.

(2) There are ***complete markets*** (a market and a price for each commodity) and ***complete and accurate information*** about each market (each participant knows the price and the characteristics — *e.g.*, the quality — of each good).

(3) There are ***no externalities***.

When we use an economic model to obtain understanding, explanation, or prediction, we don't expect the model to provide any of these perfectly. But we expect the model to give us better explanations or predictions if the above assumptions are approximately satisfied when we apply the model. While we don't expect the actual economy to be precisely in equilibrium, we expect the model's equilibrium to be "near" the economy's actual state. Therefore there's an additional assumption that's important:

(4) The underlying data (preferences, technology, *etc.*) are not changing too rapidly, so that stability issues are not very important and we can expect to actually be "near" an equilibrium.

The remainder of the course will consist almost entirely of discarding one after another of the first three assumptions: first developing the model when participants aren't price-takers; then when there are incomplete markets (in particular, incomplete markets for dealing with uncertainty and time); and then when there are externalities. We begin by studying the ***core***, a notion of bargaining equilibrium in which the participants don't take prices as given. We'll also make brief mention of the fourth assumption as well.

The Core:
Edgeworth's "Recontracting" or Bargaining Equilibrium

If there are only two consumers in the market, each one's potential to influence the prices is obvious. It's therefore unsatisfactory to assume, as the Walrasian equilibrium does, that each consumer ignores this potential and simply behaves as a price-taker. Edgeworth asked what the outcome would be if we *don't* assume that markets and prices are used. Suppose we assume only that resources are privately owned and controlled and that the traders will "bargain" with one another. Can we say anything about which allocations will or will not occur? In other words, which allocations could be viable *equilbria* of a bargaining process? Edgeworth analyzed this question for two goods and an arbitrary number of traders. Debreu & Scarf generalized to any number of goods, and others have generalized the Debreu & Scarf result.

Just as we abstracted away from the dynamics of changing prices when we analyzed market equilibrium, here we abstract away from the dynamic process of bargaining, focusing only on those situations from which no (further) bargaining will occur. And of course we begin with the simplest case, the "Edgeworth Box" situation: exchange only (no production), two goods, and two people.

At what allocations will the traders agree to trade, thereby moving to a different allocation? A first answer: they will trade if they can both gain from trading — *i.e.*, if a Pareto improvement exists. Therefore, an "equilibrium" must at least be Pareto efficient. Are *all* Pareto efficient allocations equilibria, so that the two notions of bargaining equilibrium and Pareto efficiency coincide? No: neither trader will agree to a proposed allocation that makes him worse off than he would have been if he had not traded — *i.e.*, to an allocation that makes him worse off than he is at the bundle he owns initially.

In the two-trader case, then, a reasonable definition of bargaining equilibrium is as follows:

An allocation $\hat{\mathbf{x}} = (\hat{\mathbf{x}}^i)_{i \in N}$ is a bargaining equilibrium for $N = \{1, 2\}$ if

(1) $\hat{\mathbf{x}}$ is Pareto efficient, and

(2) For each i: $u^i(\hat{\mathbf{x}}^i) \geqq u^i(\mathring{\mathbf{x}}^i)$.

An allocation that satisfies (2) is said to be *individually acceptable* or *individually rational*.

The bargaining equilibria in the two-trader case are exactly the allocations that Edgeworth called the "recontracting" equilibria. Today we call them the *core* allocations. In the Edgeworth Box they form the "contract curve." (Many economists use the term "contract curve" to mean the locus of *all* the Pareto allocations in the box, typically a much larger set.) Note that in this two-person case, with increasing preferences, every Walrasian equilibrium allocation is in the core (because a Walrasian allocation is Pareto efficient and individually rational), but almost all core allocations are *not* Walrasian equilibria for the given distribution of initial ownership.

Example 1: On the following page is a simple example in which the core of a two-person economy is determined.

In the definition of the two-person core given above, we can paraphrase Conditions (1) and (2) as follows:

(1) There is no way for the two individuals together to improve upon the allocation $\hat{\mathbf{x}}$, and
(2) There is no way for either individual, by himself, to unilaterally improve upon $\hat{\mathbf{x}}$.

Now suppose there are n consumers, indexed by $i \in N = \{1, \ldots, n\}$, with utility functions u^i and initial bundles $\mathring{\mathbf{x}}^i$. We say that the *core* is the set of all allocations that are feasible and cannot be improved upon by *any* "coalition" of members of N.

Definition 1: A *coalition* is a nonempty subset of N. Let $\mathbf{x} = (\mathbf{x}^i)_{i \in N} \in \mathbb{R}^{nl}_+$ be an allocation. A coalition S can *unilaterally improve upon* $\mathbf{x}$ if there is an allocation to S — say, $(\tilde{\mathbf{x}}^i)_{i \in S}$ — that is both

(a) feasible for S: $\sum_{i \in S} \tilde{\mathbf{x}}^i \leqq \sum_{i \in S} \mathring{\mathbf{x}}^i$ and
(b) a Pareto improvement for S: $u_i(\tilde{\mathbf{x}}^i) \geqq u^i(\mathbf{x}^i)$ for all $i \in S$, and $u_i(\tilde{\mathbf{x}}^i) > u^i(\mathbf{x}^i)$ for some i.

Definition 2: The *core* of an economy is the set of all allocations that are feasible and that cannot be improved upon by any coalition.

In the language of Definition 1, the Pareto allocations are the feasible allocations that can't be improved upon by the coalition of *all* individuals, and the individually rational allocations are the ones that can't be unilaterally improved upon by any of the *one-person* coalitions. Therefore all core allocations are both Pareto efficient and individually rational, just as in the two-person case. But as Example 2 below shows, when $n > 2$ there are generally allocations that are both Pareto efficient and individually rational but which are nevertheless *not* in the core — because they can be improved upon by some intermediate-sized coalition, one that includes more than one person but not everyone.

A Simple Core Example

$u_A(x_A, y_A) = x_A y_A$ $\qquad$ $u_B(x_B, y_B) = x_B y_B$

$(\mathring{x}_A, \mathring{y}_A) = (10, 90)$ $\qquad$ $(\mathring{x}_B, \mathring{y}_B) = (90, 10)$

$\mathring{u}_A = 900$ $\qquad$ $\mathring{u}_B = 900$

The Core:

$MRS_A = MRS_B$ $\qquad$ $u_A \geqq 900$ $\qquad$ $u_B \geqq 900$

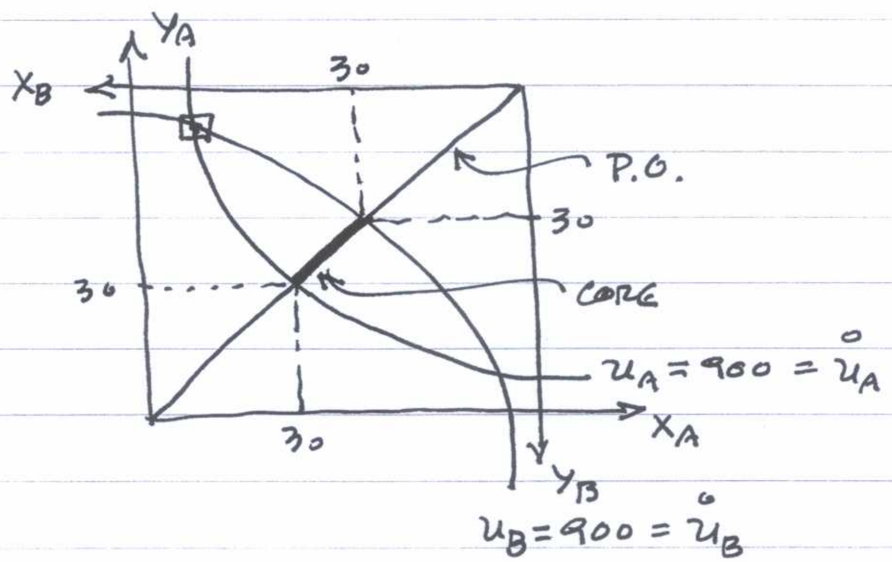
$\Downarrow$

$x_A = y_A, \therefore u_A = x_A^2$ $\qquad$ $\therefore x_A^2 \geqq 900, \; x_A \geqq 30$

$x_B = y_B, \therefore u_B = x_B^2$ $\qquad$ $\therefore x_B^2 \geqq 900, \; x_B \geqq 30$

Summarizing: $x_A = y_A$; $x_A + x_B = 100$, $y_A + y_B = 100$;

$30 \leqq x_A \leqq 70$.



Example 2: This example shows that when $n > 2$ the requirement that *no* coalition be able to improve upon $(\hat{\mathbf{x}}^i)_1^n$ is generally a stronger requirement than merely requiring Pareto efficiency ("N cannot improve") and individual rationality ("no {i} can improve").

$$N = \{1,2,3\} \qquad \forall i \in N : u^i(\mathbf{x}^i) = x_1^i x_2^i$$

$$\begin{array}{rcl l} \mathring{\mathbf{x}}^1 & = & (19,1) & \therefore \mathring{u}^1 = 19 \\ \mathring{\mathbf{x}}^2 & = & (1,19) & \therefore \mathring{u}^2 = 19 \\ \mathring{\mathbf{x}}^3 & = & (10,10) & \therefore \mathring{u}^3 = 100 \\ \hline \sum \mathring{\mathbf{x}}^i & = & (30,30) & \end{array}$$

The following allocation $(\hat{\mathbf{x}}^i)_1^3$ is Pareto efficient and individually rational:

$$\begin{array}{rcccl l} \hat{\mathbf{x}}^1 & = & \hat{\mathbf{x}}^2 & = & (9,9) & \therefore \hat{u}^1 = \hat{u}^2 = 81 \\ \hat{\mathbf{x}}^3 & = & & & (12,12) & \therefore \hat{u}^3 = 144 \\ \hline \sum \hat{\mathbf{x}}^i & = & & & (30,30) & \end{array}$$

But the coalition $S = \{1,2\}$ can unilaterally improve upon $(\hat{\mathbf{x}}^i)_1^3$ as follows:

$$\tilde{\mathbf{x}}^1 = \tilde{\mathbf{x}}^2 = (10,10) \qquad \therefore \tilde{u}^1 = \tilde{u}^2 = 100 > 81.$$

Therefore $(\hat{\mathbf{x}}^i)_1^3$ is not in the core.

Exercise: Exercise #5.1 in the Exercise Book asks you to find all the core allocations for this example.

Walrasian Equilibria are in the Core

In our examples, we've noted in every case that the Walrasian equilibrium allocation has been in the core — *i.e.*, it's been one of the possible "bargaining equilibria" of the economy. The theorem on the following page establishes that this is true under very general conditions: the consumers' preferences simply need to be locally nonsatiated. But this shouldn't be surprising: local nonsatiation is also the only assumption needed to establish the First Welfare Theorem, which says that if you can make a Pareto improvement on a proposed allocation, then the proposed allocation can't be the outcome of a Walrasian equilibrium. In fact, that's exactly the way our proof proceeded: we assumed that the proposed equilibrium allocation is not Pareto (*i.e.*, it can be improved upon), and then showed that the assumed improvement could not be feasible, a contradiction — the assumed improvement can't actually be accomplished.

To establish that a Walrasian allocation is more than just Pareto optimal, and in fact is actually in the core, we proceed in exactly the same way: we assume that the proposed allocation can be improved upon by *some* coalition — but not necessarily by the coalition consisting of all the traders — and show in the same way as before that the improvement could not be feasible for the coalition, using just its own resources. In other words, we show that an assumed improvement by *any* coalition can't actually be accomplished with the resources available to it, and therefore the proposal *is* in the core.

Note that the proof is identical to our proof of the First Welfare Theorem except that an arbitrary coalition S replaces the specific coalition N consisting of all traders, just as the above discussion suggests.

Theorem: Let $(\hat{p}, (\hat{x}^i)_N)$ be a competitive equilibrium for an economy $E = ((u^i, \mathring{x}^i))_N$, and assume that $\hat{p} \in \mathbb{R}^\ell_+$. If each u^i is locally nonsatiated (LNS), then $(\hat{x}^i)_N$ is in the core of E.

Proof:

Suppose that $(\hat{x}^i)_N$ is not in the core — i.e., there is a coalition S that can unilaterally improve upon $(\hat{x}^i)_N$ via an allocation $(\tilde{x}^i)_S$:

(a) $\sum_{i \in S} \tilde{x}^i \leqq \sum_{i \in S} \mathring{x}^i$

(b1) $\forall i \in S: u^i(\tilde{x}^i) \geqq u^i(\hat{x}^i)$,

(b2) $\exists i \in S: u^i(\tilde{x}^i) > u^i(\hat{x}^i)$.

Because $(\hat{p}, (\hat{x}^i)_N)$ is a competitive equilibrium, each $\hat{x}^i$ maximizes u^i on the budget set $\{x^i \in \mathbb{R}^\ell_+ \mid \hat{p}\cdot x^i \leqq \hat{p}\cdot\mathring{x}^i\}$. Therefore, (b2) ensures that

(c2) $\exists i \in S: \hat{p}\cdot\tilde{x}^i > \hat{p}\cdot\mathring{x}^i$,

and (b1) ensures that

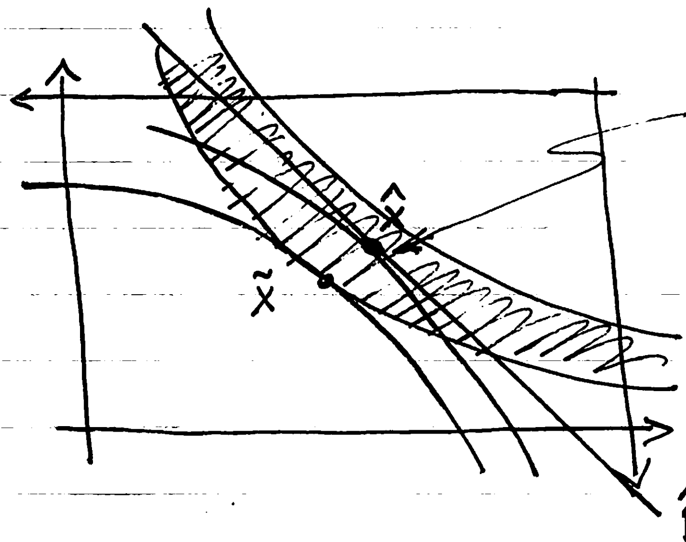
(c1) $\forall i \in S: \hat{p}\cdot\tilde{x}^i \geqq \hat{p}\cdot\mathring{x}^i$,

Duality [utility max. $\Rightarrow$ expenditure min.] yields $\hat{p}\cdot\tilde{x}^i \geqq \hat{p}\cdot\hat{x}^i$, but that's not enough if we don't also know that $\hat{p}\cdot\hat{x}^i \geqq \hat{p}\cdot\mathring{x}^i$.

as follows: If $\hat{p}\cdot\tilde{x}^i < \hat{p}\cdot\mathring{x}^i$, there would be a neighborhood η of $\tilde{x}^i$ for which $x^i \in \eta \Rightarrow \hat{p}\cdot x^i < \hat{p}\cdot\mathring{x}^i$, and by LNS such a nbd. contains an x^i that satisfies $u^i(x^i) > u^i(\tilde{x}^i) \geqq u^i(\hat{x}^i)$, which is inconsistent with $\hat{x}^i$ maximizing u^i on the budget set. Summing the inequalities in (c1) and (c2) over S, we have $\sum_S \hat{p}\cdot\tilde{x}^i > \sum_S \hat{p}\cdot\mathring{x}^i$ — i.e., $\hat{p}\cdot\sum_S \tilde{x}^i > \hat{p}\cdot\sum_S \mathring{x}^i$. Since $\hat{p} \in \mathbb{R}^\ell_+$, this yields $\sum_S \tilde{x}^i_k > \sum_S \mathring{x}^i_k$ for some k, which contradicts (a). ||

COUNTEREXAMPLE:

(TO SHOW THAT LNS IS ESSENTIAL)



THIS A WALRASIAN EQUIL'M, BUT $(\tilde{x}^i)_N$ IS A PARETO IMPROVEMENT; $\therefore (\hat{x}^i)_N$ IS NOT IN THE CORE.

THIS IS THE SAME EXAMPLE (MR. 1 HAS A THICK I-CURVE) AS FOR THE FIRST WELFARE THEOREM. NOTICE THAT THE PROOF GIVEN ABOVE IS ALSO VIRTUALLY THE SAME AS THE PROOF OF THE FIRST WELFARE THEOREM.

The Utility Frontier

Any allocation $(\mathbf{x}^i)_1^n$ to a set $N = \{1, \ldots, n\}$ of individuals with utility functions $u^1(\cdot), \ldots, u^n(\cdot)$ yields a profile $(u_1, \ldots, u_n)$ of resulting utility levels, as depicted in Figure 1 for the case $n = 2$. (Throughout this set of notes, in order to distinguish between utility *functions* and utility *levels*, I'll use superscripts for the functions and subscripts for the resulting levels, as I've done in the preceding sentence and in Figure 1.) Let's formally define the function that accomplishes this:

$$U : \mathbb{R}_+^{n\ell} \to \mathbb{R}^n \text{ is defined by } U\left((\mathbf{x}^i)_N\right) = \left(u^1(\mathbf{x}^1), \ldots, u^n(\mathbf{x}^n)\right) \tag{$*$}$$

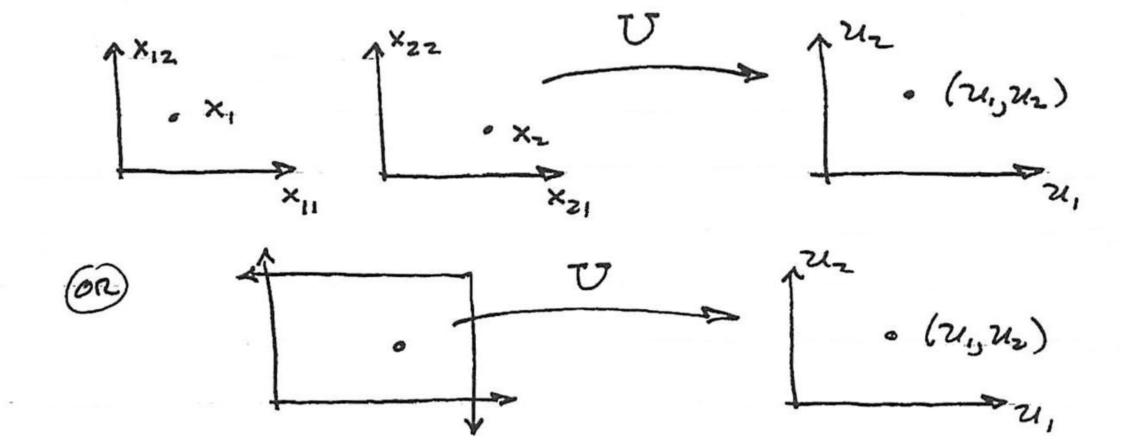


Figure 1

Let $\mathcal{F}$ denote the set of feasible allocations — *i.e.*, those that satisfy $\sum_1^n \mathbf{x}^i \leqq \mathring{\mathbf{x}}$. The set of **feasible utility profiles** is the image under U of the set of all feasible allocations, *i.e.*, $U(\mathcal{F})$:

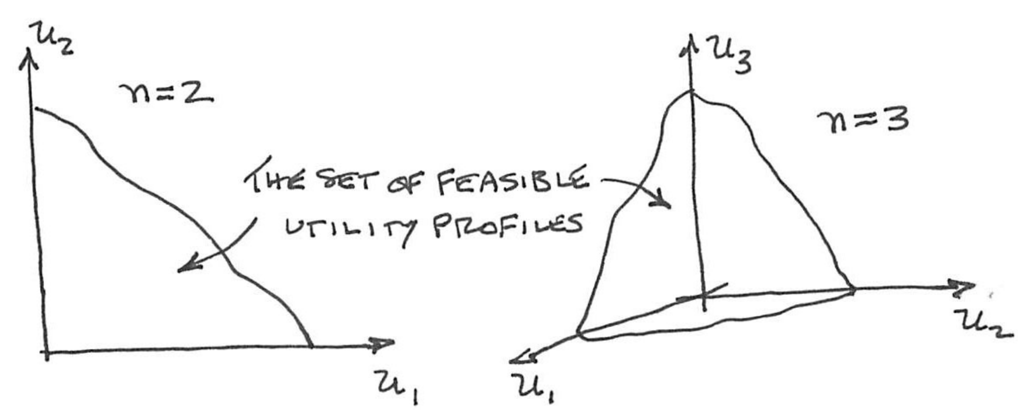


Figure 2

The Pareto efficient allocations are clearly the ones that get mapped by U to the "northeast" part of the boundary of the set of feasible utility profiles. (More accurately, to those points $\mathbf{u}$ on the boundary of $U(\mathcal{F})$ for which there are no other points in $U(\mathcal{F})$ lying to the northeast). This northeast part of the set $U(\mathcal{F})$ is called the **utility frontier**, which we'll denote by UF. It consists of the utility profiles $\mathbf{u} = (u_1, \ldots, u_n)$ that are maximal in $U(\mathcal{F})$ with respect to the preorder $\geqq$ on $\mathbb{R}^n$:

$\mathbf{u} = (u_1, \ldots, u_n) \in$ UF if and only if

$\mathbf{u} \in U(\mathcal{F})$ and there is no $\mathbf{u}' \in U(\mathcal{F})$ that satisfies $\forall i : u'_i \geqq u_i \;\&\; \exists i : u'_i > u_i$.

Equivalently, UF is the image under U of the set of Pareto allocations:

UF $= U(\mathcal{P})$, where $\mathcal{P}$ is the set of Pareto allocations in $\mathbb{R}^{n\ell}_+$.

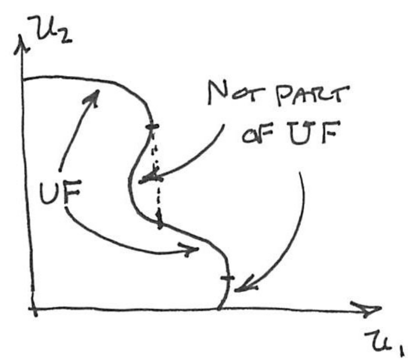


Figure 3

Note that the alternatives over which the individuals have utility functions needn't be allocations: we could replace the set $\mathbb{R}^{n\ell}_+$ of allocations with an arbitrary set X of alternatives x, and $(*)$ would become

$$U : X \to \mathbb{R}^n \text{ is defined by } U(x) = \big(u^1(x), \ldots, u^n(x)\big)$$

Figure 2 would still look the same: it would be $U(X)$, or $U(\mathcal{F})$, the image under U of either X or $\mathcal{F}$; and Figure 3 would be the same, the image under U of the set of Pareto efficient alternatives.

The utility frontier is a surface in $\mathbb{R}^n$, and it could be expressed as the set of profiles $(u_1, \ldots, u_n)$ that satisfy the equation $h(u_1, \ldots, u_n) = 0$ for some function h, or

$$u_1 = g(u_2, \ldots, u_n) \qquad (**)$$

for some function g. In the equation $(**)$, the function g tells us, for given utility levels $u_2, \ldots, u_n$ for $n - 1$ individuals, what is the maximum utility level u_1 that's feasible for the remaining individual. In other words, g is the *value function* for the problem (P-Max), in which the utility levels $u_2, \ldots, u_n$ are parameters and we solve for the allocation $(\mathbf{x}^i)_1^n$ in which $\mathbf{x}^1$ maximizes $u^1(\cdot)$ subject to all other individuals $i = 2, \ldots, n$ receiving at least the utility level u_i (recall that we're using u^i to denote utility *functions* and u_i to denote utility *levels*!):

$$\begin{aligned} &\max_{(x_k^i) \in \mathbb{R}_+^{nl}} u^1(\mathbf{x}^1) \\ \text{subject to} \quad & x_k^i \geqq 0, \quad i = 1, ..., n, \quad k = 1, ..., l \\ & \sum_{i=1}^{n} x_k^i \leqq \mathring{x}_k, \quad k = 1, ..., l \\ & u^i(\mathbf{x}^i) \geqq u_i, \quad i = 2, ..., n. \end{aligned} \tag{P-Max}$$

The Solution Function and the Value Function for a Maximization Problem

Consider the maximization problem

$$\max_x f(x; \alpha) \text{ subject to } G(x; \alpha) \leqq \mathbf{0}. \tag{$\mathbf{P}$}$$

Note that we're maximizing over x and not over α — x is a variable in the problem (typically a vector or n-tuple of variables) and α is a parameter (typically a vector or m-tuple of parameters). The parameters may appear in the objective function and/or the constraints, if there are any constraints. We associate the following two functions with the maximization problem (**P**):

the **solution function**: $x = x(\alpha)$, and

the **value function**: $v(\alpha) := f(x(\alpha))$.

The solution function gives the solution x as a function of the parameters; the value function gives the value of the objective function as a function of the parameters.

Example 1: The consumer maximization problem (CMP) in demand theory,

$$\max_{\mathbf{x} \in \mathbb{R}_+^\ell} u(\mathbf{x}) \text{ subject to } \mathbf{p} \cdot \mathbf{x} \leqq w.$$

Here α is the $(\ell + 1)$-tuple $(\mathbf{p}; w)$ consisting of the price-list $\mathbf{p}$ and the consumer's wealth w.

The solution function is the consumer's demand function $\mathbf{x}(\mathbf{p}; w)$.

The value function is the consumer's indirect utility function $v(\mathbf{p}; w) = u(\mathbf{x}(\mathbf{p}; w))$.

Example 2: The firm's cost-minimization (*i.e.*, expenditure-minimization) problem,

$$\min_{\mathbf{x}\in\mathbb{R}^\ell_+} E(\mathbf{x};\mathbf{w}) = \mathbf{w}\cdot\mathbf{x} \;\text{ subject to }\; F(\mathbf{x}) \geqq y.$$

Here F is the firm's production function; $\mathbf{x}$ is the ℓ-tuple of input levels that will be employed; $E(\mathbf{x};\mathbf{w})$ is the resulting expenditure the firm will incur; and α is the $(\ell+1)$-tuple $(y;\mathbf{w})$ consisting of the proposed level of output, y, and the ℓ-tuple $\mathbf{w}$ of input prices.

The solution function is the firm's input demand function $\mathbf{x}(y;\mathbf{w})$.

The value function is the firm's cost function $C(y;\mathbf{w}) = E(\mathbf{x}(y;\mathbf{w});\mathbf{w})$.

Example 3: The Pareto problem (P-Max),

$$\max_{\mathbf{x}\in\mathcal{F}} u^1(\mathbf{x}^1) \;\text{ subject to }\; u^2(\mathbf{x}^2) \geqq u_2\,,\;\ldots\;,\; u^n(\mathbf{x}^n) \geqq u_n,$$

where $\mathcal{F}$ is the feasible set $\{\mathbf{x}\in\mathbb{R}^{n\ell}_+ \mid \sum_1^n \mathbf{x}^i \leqq \mathring{\mathbf{x}}\}$. Here α is the $(n-1)$-tuple of utility levels $u_2,\ldots,u_n$.

The solution function is $\mathbf{x}(u_2,\ldots,u_n)$, which gives the Pareto allocation as a function of the utility levels $u_2,\ldots,u_n$.

The value function is $u^1(\mathbf{x}(u_2,\ldots,u_n))$, which gives the maximum attainable utility level u_1 as a function of the utility levels $u_2,\ldots,u_n$.

The value function therefore describes the utility frontier for the economy $((u^i)_1^n,\mathring{\mathbf{x}})$, as depicted in Figure 2.

EXAMPLE:

$N=\{1,\dots,n\}$; $u_i(x_i,y_i)=x_i y_i$, $\forall i\in N$;

TOTAL ENDOWMENT IS $(\mathring{x},\mathring{y})$.

PARETO EFFICIENCY REQUIRES THAT, FOR SOME NUMBER r:

$$\frac{y_1}{x_1}=\frac{y_2}{x_2}=\dots=\frac{y_n}{x_n}=r; \quad \text{i.e., } y_i=rx_i,\ \forall i\in N.$$

$$\therefore\ \mathring{y}=r\mathring{x} \quad \left[\mathring{y}=\Sigma y_i=\Sigma r x_i=r\Sigma x_i=r\mathring{x}\right],$$

$$\text{i.e., } \boxed{r=\frac{\mathring{y}}{\mathring{x}}}$$

AT ANY EFFICIENT ALLOCATION, THEN, WE MUST HAVE, $\forall i\in N$:

$$u_i(x_i,y_i)=x_i y_i=(x_i)(rx_i)=rx_i^2;$$

$$\text{i.e., } \sqrt{u_i}=\sqrt{r}\,x_i.$$

$$\therefore\ \sum_{i\in N}\sqrt{u_i}=\sqrt{r}\sum_{i\in N}x_i=\sqrt{r}\,\mathring{x}.$$

$$\therefore\ \sum_{i\in N}\sqrt{u_i}=\sqrt{r}\,\mathring{x}=\frac{\sqrt{\mathring{y}}}{\sqrt{\mathring{x}}}\,\mathring{x}=\sqrt{\mathring{x}\mathring{y}}.$$

IN OTHER WORDS, THE UTILITY FRONTIER IS THE EQUATION

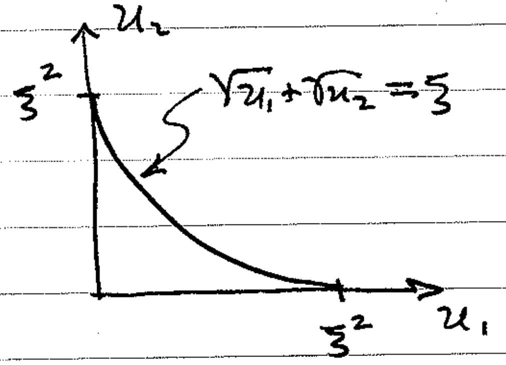
$$\sum_{i\in N}\sqrt{u_i}=\sqrt{\mathring{x}\mathring{y}},$$

OR ITS GRAPH IN $\mathbb{R}^n$.

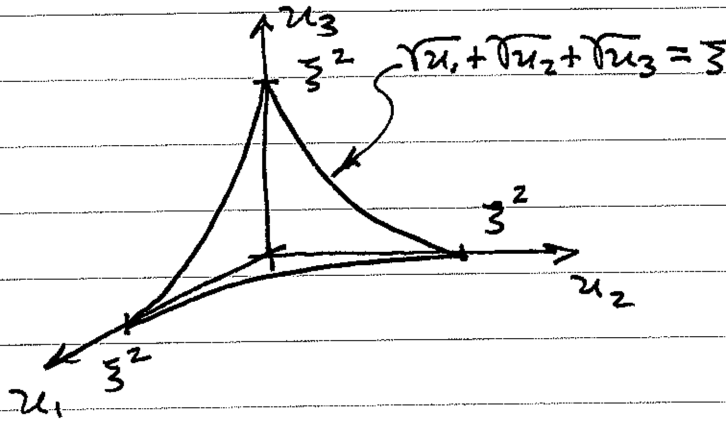
DANGER: THE UTILITY FRONTIER HAS THIS FORM <u>IN THIS EXAMPLE</u>, WHERE ALL UTILITY FUNCTIONS ARE OF THE FORM $u(x,y)=xy$.

Let $\xi := \sqrt{\mathring{x}\mathring{y}}$.

$n = 2:$



$n = 3:$



EXAMPLE: (THE UTILITY FRONTIER AND THE CORE)

$N = \{1,2,3\}$; $u_i(x_{i1}, x_{i2}) = x_{i1} \cdot x_{i2}$, $i = 1,2,3.$

$\mathring{x}_1 = \mathring{x}_2 = (30,0)$; $\mathring{x}_3 = (0,60)$.

PROPOSAL: $\hat{x}_i = (20,20)$, $i=1,2,3$. $u_i(\hat{x}_i) = 400$, $i=1,2,3$.

CLEARLY, $(\hat{x}_i)_N$ IS PARETO EFFICIENT AND INDIVIDUALLY ACCEPTABLE.

BUT $\{1,3\}$ CAN IMPROVE UPON $(\hat{x}_i)_N$ VIA $(\tilde{x}_i)_{\{1,3\}}$,

WHERE $\tilde{x}_1 = \tilde{x}_3 = (15,30)$:

WE HAVE $\tilde{x}_1 + \tilde{x}_3 = (30,60) = \mathring{x}_1 + \mathring{x}_3$ AND $u_1(\tilde{x}_1) = u_2(\tilde{x}_2) = 450.$

THE COALITION $\{2,3\}$ COULD IMPROVE IN THE SAME WAY.

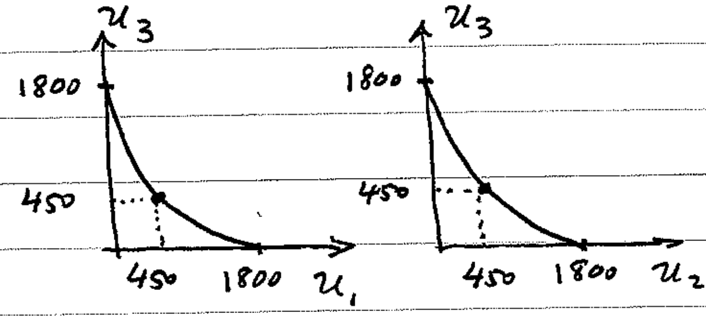
IN FACT, IT IS CLEAR THAT UNLESS A PROPOSAL $(x_i)_N$ GIVES BOTH $u_1 \geqq 450$ AND $u_2 \geqq 450$, OR ELSE $u_3 \geqq 450$, THEN EITHER $\{1,3\}$ OR $\{2,3\}$ WILL BE ABLE TO UNILATERALLY IMPROVE UPON $(x_i)_N$: ANY PROPOSAL THAT $u_3 < 450$ AND EITHER $u_1 < 450$ OR $u_2 < 450$ CAN BE IMPROVED UPON BY $\{1,3\}$ OR $\{2,3\}$ AS ABOVE.

IN FACT, THE UTILITY FRONTIERS FOR $\{1,3\}$ AND $\{2,3\}$ ARE $\sqrt{u_1} + \sqrt{u_3} = \sqrt{(30)(60)} = \sqrt{1800} = 30\sqrt{2}$

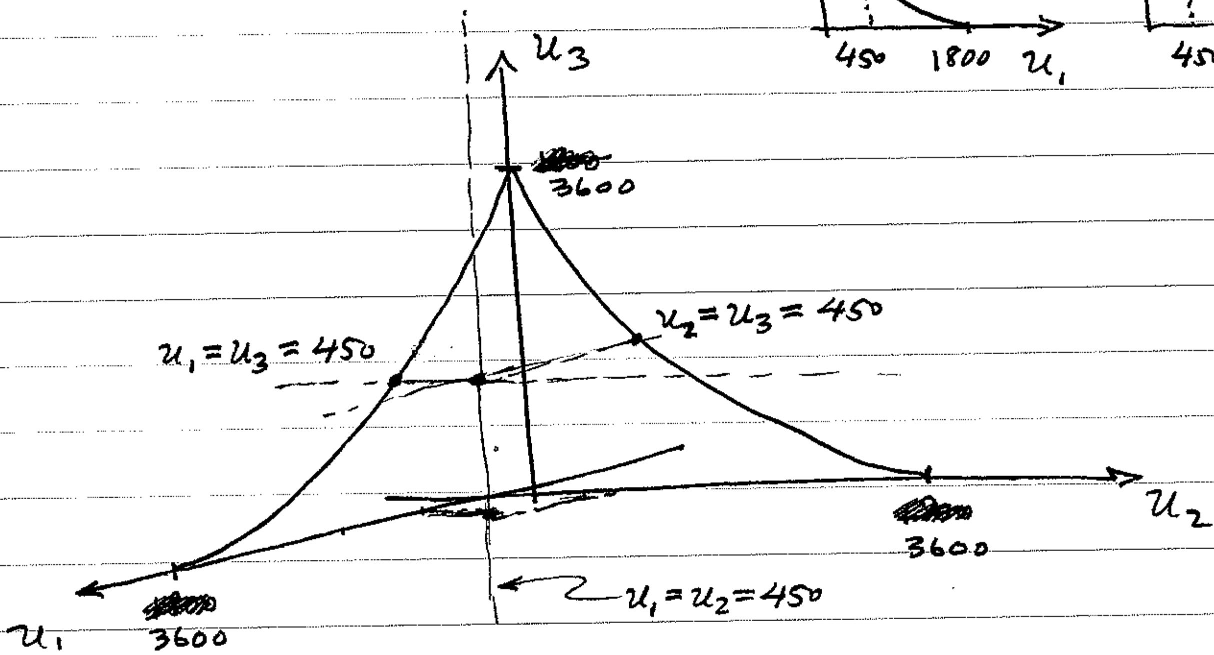
AND $\sqrt{u_2} + \sqrt{u_3} = \sqrt{(30)(60)} = \sqrt{1800} = 30\sqrt{2}$.

SINCE PARETO EFFICIENCY IN THIS EXAMPLE REQUIRES $x_{i1} = x_{i2} = z_i$, SAY, FOR $i=1,2,3$, WE HAVE

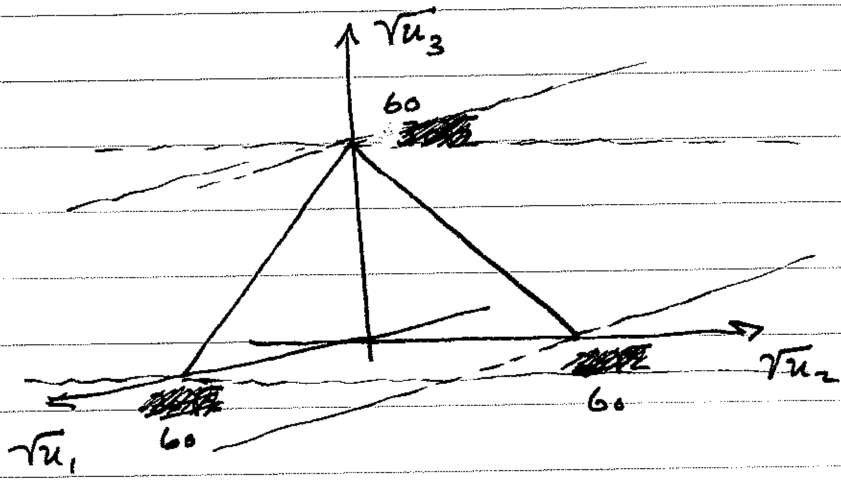
$$z_1 + z_3 \geqq 30\sqrt{2} \approx 42.4 \text{ AND } z_2 + z_3 \geqq 30\sqrt{2} \approx 42.4.$$



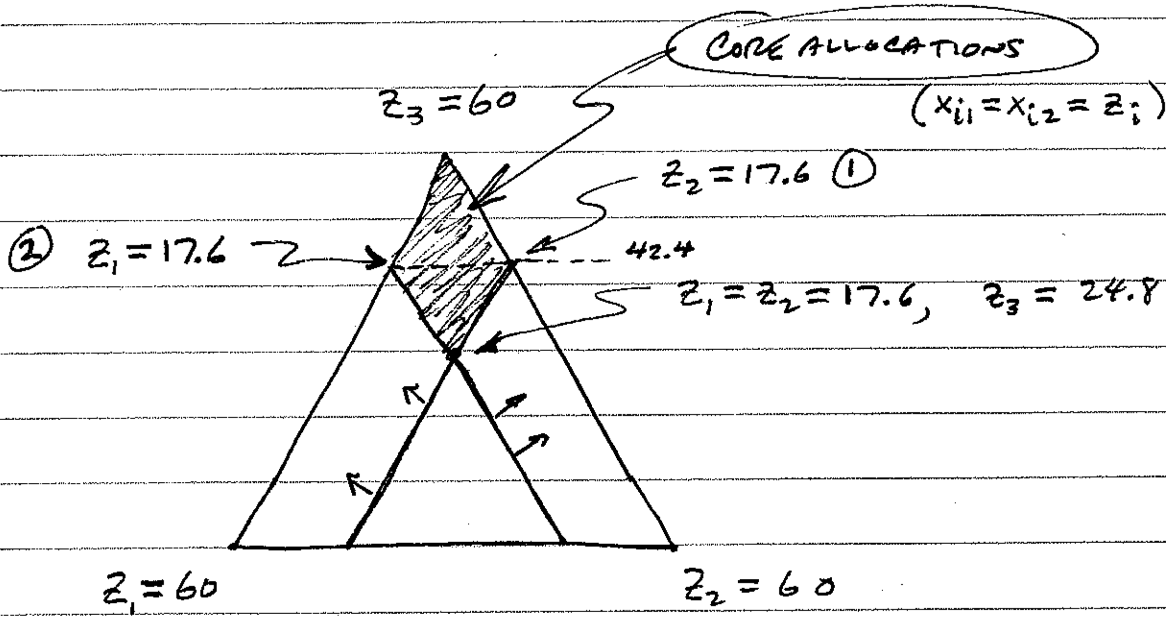
u_3
1800
450
450
1800
u_1
u_3
1800
450
450
1800
u_2



u_3
3600
$u_2 = u_3 = 450$
$u_1 = u_3 = 450$
u_2
3600
$u_1 = u_2 = 450$
u_1
3600



$\sqrt{u_3}$
60
$\sqrt{u_2}$
60
$\sqrt{u_1}$
60



① $z_1 + z_3 \geqq 42.4$ i.e, $z_2 \leqq 17.6$

② $z_2 + z_3 \geqq 42.4$ i.e, $z_1 \leqq 17.6$

$\therefore$ $z = (20, 20, 20)$ IS <u>NOT</u> IN THE CORE

A Core Example with Four Consumers

In this example we're going to have four consumers, indexed by $i \in N = \{1, 2, 3, 4\}$. The two "odd" consumers ($i = 1, 3$) will be identical and the two "even" consumers ($i = 2, 4$) will be identical. The example will show how it is that adding more consumers to the economy can remove some allocations from the core, in a certain sense, thus shrinking the core to a smaller set of allocations.

Each consumer in the example has the same utility function, $u(x, y) = xy$. The two odd consumers are each endowed with one unit of the y-good and none of the x-good, and each of the even consumers is endowed with one unit of the x-good and none of the y-good:

$$(\mathring{x}^1, \mathring{y}^1) = (\mathring{x}^3, \mathring{y}^3) = (0, 1) \quad \text{and} \quad (\mathring{x}^2, \mathring{y}^2) = (\mathring{x}^4, \mathring{y}^4) = (1, 0).$$

The allocations for this economy are in $\mathbb{R}^8_+$, so we're not going to get very far trying to use an Edgeworth box-type diagram, which will be six-dimensional. But we can exploit the fact that there are only two distinct "types" of consumer, the odds and the evens, by first analyzing an economy in which there is only *one* consumer of each type, say just $i = 1$ and $i = 2$. In this two-person economy it's easy to see that the Pareto allocations are the ones in which $x^1 = y^1$ and $x^2 = y^2$ — *i.e.*, the diagonal of the Edgeworth box (EB). The unique Walrasian equilibrium (WE) has $p_x = p_y$ and each consumer receives the bundle $(x^i, y^i) = (\frac{1}{2}, \frac{1}{2})$. See Figure 1.

Now returning to the four-person economy, the Walrasian equilibrium still has $p_x = p_y$, and each consumer still receives the bundle $(x^i, y^i) = (\frac{1}{2}, \frac{1}{2})$. The Pareto allocations are still the ones in which $x^i = y^i$ for each consumer. So the Edgeworth box is still somewhat useful here: in the WE, each consumer of a given type receives the bundle he would receive in the Edgeworth box economy, and the Pareto set is similar to the Pareto set in the EB economy.

What about the core in the two economies? In the two-person economy, with one consumer of each type, the core is the entire diagonal of the box. In particular, the lower-left-corner allocation, in which $(x^1, y^1) = (0, 0)$ and $(x^2, y^2) = (1, 1)$, is in the core. In the corresponding four-person allocation, both odd consumers receive $(0, 0)$ and both even consumers receive $(1, 1)$. We'll denote this allocation by $(\widehat{x}^i, \widehat{y}^i)_N$, where $N = \{1, 2, 3, 4\}$. See Figure 1. Is $(\widehat{x}^i, \widehat{y}^i)_N$ in the core of the four-person economy? Let's figure it out.

(The corner allocation, in which the odd consumers each get $(0, 0)$, may seem kind of extreme. Before we're done we'll see that the argument we're going to develop applies to plenty of not-so-extreme allocations too. In fact, to all but the WE allocation, where each $(x^i, y^i) = (\frac{1}{2}, \frac{1}{2})$!)

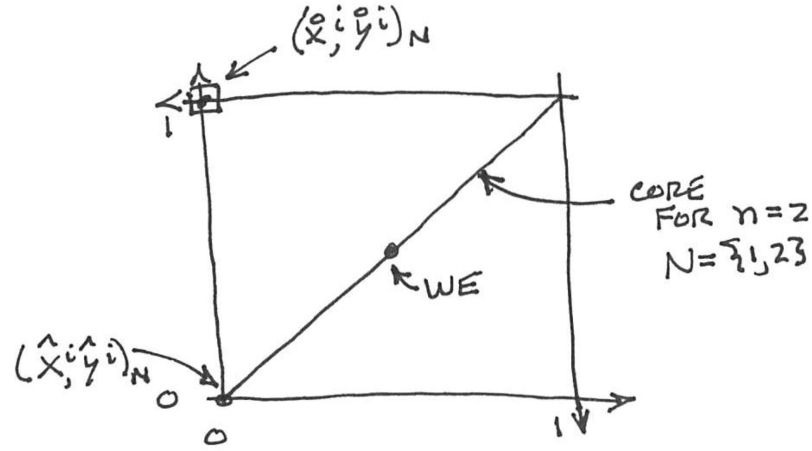


Figure 1

How can we show that a given allocation, such as $(\widehat{x}^i, \widehat{y}^i)_N$, is in the core, or alternatively that it's *not* in the core? If we can find just one coalition S that can unilaterally improve on $(\widehat{x}^i, \widehat{y}^i)_N$, then it's not in the core. Conversely, in order to show that the allocation *is* in the core, we have to show that none of the fifteen possible coalitions can unilaterally improve on it.

It's clear that the four-person "coalition of the whole," $S = N$, can't improve on $(\widehat{x}^i, \widehat{y}^i)_N$, because $(\widehat{x}^i, \widehat{y}^i)_N$ is Pareto efficient. It's also clear that none of the one-person coalitions $\{1\}, \{2\}, \{3\}$, or $\{4\}$ can improve, because each consumer is receiving at least as much utility in $(\widehat{x}^i, \widehat{y}^i)_N$ as she receives by just consuming her initial bundle.

For most of the remaining coalitions (*i.e.*, the two- and three-person coalitions), it's also not too difficult to show that the coalition can't improve on $(\widehat{x}^i, \widehat{y}^i)_N$. For example, the coalition $S = \{1, 3\}$ consisting of just the two odd consumers owns two units of the y-good but none of the x-good, so even though each of these two consumers receives $u_i = 0$ in $(\widehat{x}^i, \widehat{y}^i)_N$, they can't do any better than that with just their own resources.

However, there's a more systematic way to determine whether coalitions can improve on a given allocation such as $(\widehat{x}^i, \widehat{y}^i)_N$: we can use the coalition's utility frontier. Recall from our Utility Frontier notes that we used the notation $\mathcal{F}$ to denote the set of feasible allocations for the set N of all the consumers — *i.e.*, the allocations that satisfy $\sum_{i \in N} \mathbf{x}^i \leqq \mathring{\mathbf{x}}$. We used $\mathcal{P}$ to denote the set of Pareto allocations, and we used U to denote the function that maps allocations $(\mathbf{x}^i)_N$ to their resulting utility profiles $(u_i)_N = U((\mathbf{x}^i)_N) = (u^1(\mathbf{x}^1), \ldots, u^n(\mathbf{x}^n))$. In the space $\mathbb{R}^n$ of utility profiles, $U(\mathcal{F})$ is the set of feasible utility profiles — the image under U of the set of feasible allocations — and $U(\mathcal{P})$ is the utility frontier, the image under U of the set of Pareto efficient allocations.

Let's replace the set N everywhere in the preceding paragraph with a coalition $S \subseteq N$, and write $\mathcal{F}_S$ for the set of allocations $(x^i)_S$ to S that are feasible for S — *i.e.*, $\sum_{i \in S} \mathbf{x}^i \leqq \mathring{\mathbf{x}}_S$ — and write $\mathcal{P}_S$ for the set of allocations to S that are Pareto efficient for S using just its own resources. Now consider an allocation $(\widehat{\mathbf{x}}^i)_N$ to N and the associated utility profile $(\widehat{u}_i)_N$. A coalition S can unilaterally improve upon $(\widehat{\mathbf{x}}^i)_N$ exactly if there is some allocation $(\mathbf{x}^i)_S$ to S that's feasible for S and a Pareto improvement (for S!) upon $(\widehat{\mathbf{x}}_i)_S$ — in other words, S can unilaterally improve if S's part of the profile $(\widehat{u}_i)_N$, namely $(\widehat{u}_i)_S$, is in $U(\mathcal{F}_S)$ but not on the utility frontier, $U(\mathcal{P}_S)$. If the utility frontier for S is described by the equation $h_S((u_i)_S) = c$ for some real number c and some strictly increasing function h_S, then S can unilaterally improve upon $(\widehat{\mathbf{x}}^i)_N$ if and only if

$$h_S((\widehat{u}_i)_S) < c\,, \tag{1}$$

In short, S can unilaterally improve if and only if $(\widehat{u}_i)_S$ lies strictly inside S's utility frontier.

Let's see how this works in our four-consumer example. In the Utility Frontier notes we saw that if a set S of consumers has a total of $\mathring{x}_S$ units of the x-good and $\mathring{y}_S$ units of the y-good, and if every one of the consumers has the utility function $u(x, y) = xy$, then the utility frontier is the set of utility profiles that satisfy the equation

$$\sum_{i \in S} \sqrt{u_i} = \sqrt{\mathring{x}_S \mathring{y}_S}\,. \tag{2}$$

Since the coalition of the whole, $S = N$, has $(\mathring{x}, \mathring{y}) = (2, 2)$, the utility frontier for N is $\sum_{i \in N} \sqrt{u_i} = 2$. The proposed allocation $(\widehat{x}^i, \widehat{y}^i)_N$, in which each odd consumer receives $(0, 0)$ and each even consumer receives $(1, 1)$, yields the utility profile $(\widehat{u}_i)_{i \in N} = (0, 1, 0, 1)$, which is *on* the utility frontier for N — which we knew it must be, because $(\widehat{x}^i, \widehat{y}^i)_N$ is Pareto efficient.

Let's look at all the other coalitions:

The one-person coalitions, $S = \{1\}, \{2\}, \{3\}, \{4\}$:

UF$_S$ is $u_i = u(\mathring{x}^i, \mathring{y}^i) = 0$. Since $\widehat{u}_i = 0$ for i odd and $\widehat{u}_i = 1$ for i even, $\widehat{u}_i$ is either on or outside the utility frontier for each i. So none of these coalitions can unilaterally improve upon $(\widehat{x}^i, \widehat{y}^i)_N$.

The two-person coalitions $S = \{1, 2\}, \{1, 4\}, \{3, 2\}, \{3, 4\}$: (See Figure 2.)

Each of these is just like the two-person Edgeworth box economy, with one odd consumer and one even, and UF$_S$ is given by $\sqrt{u_{odd}} + \sqrt{u_{even}} = 1$. Since $\widehat{u}_{odd} = 0$ and $\widehat{u}_{even} = 1$, we have $(\widehat{u}_{odd}, \widehat{u}_{even}) = (0, 1)$, which is on the utility frontier for S. So none of these coalitions can unilaterally improve upon $(\widehat{x}^i, \widehat{y}^i)_N$.

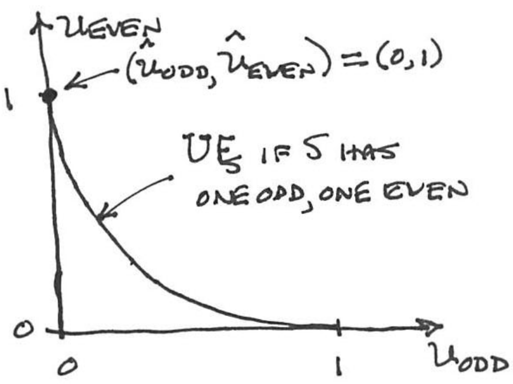


Figure 2

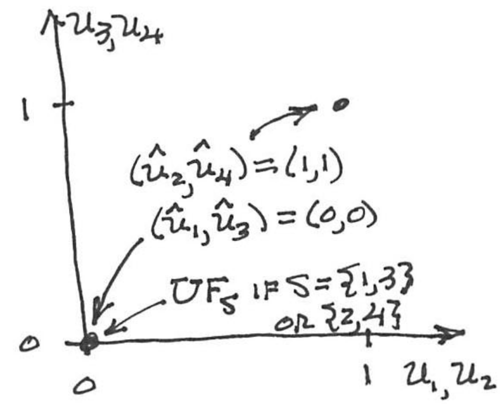


Figure 3

The two-person coalitions $S = \{1, 3\}$ and $S = \{2, 4\}$: (See Figure 3.)

In each of these two cases the coalition has none of one of the two goods, so its utility frontier is given by $\sum_{i \in S} \sqrt{u_i} = 0$, and the utility frontier is therefore the singleton $\{(0, 0)\}$. For $S = \{1, 3\}$, the utility profile $(\widehat{u}_1, \widehat{u}_3)$ is $(0, 0)$, which is on the utility frontier. For $S = \{2, 4\}$, the utility profile $(\widehat{u}_2, \widehat{u}_4)$ is $(1, 1)$, which is outside the utility frontier. So neither of these coalitions can unilaterally improve upon $(\widehat{x}^i, \widehat{y}^i)_N$.

The three-person coalitions $S = \{1, 2, 4\}$ and $S = \{3, 2, 4\}$: (See Figure 4.)

Each of these coalitions has one odd consumer and two even consumers; we'll work with $S = \{1, 2, 4\}$. We have $(\mathring{x}_S, \mathring{y}_S) = (2, 1)$, so the utility frontier for S is given by the equation $\sqrt{u_1} + \sqrt{u_2} + \sqrt{u_4} = \sqrt{2}$. The utility profile $(\widehat{u}_1, \widehat{u}_2, \widehat{u}_4) = (0, 1, 1)$ is outside this utility frontier. The same is true for $S = \{3, 2, 4\}$, so neither of these coalitions can unilaterally improve upon $(\widehat{x}^i, \widehat{y}^i)_N$.

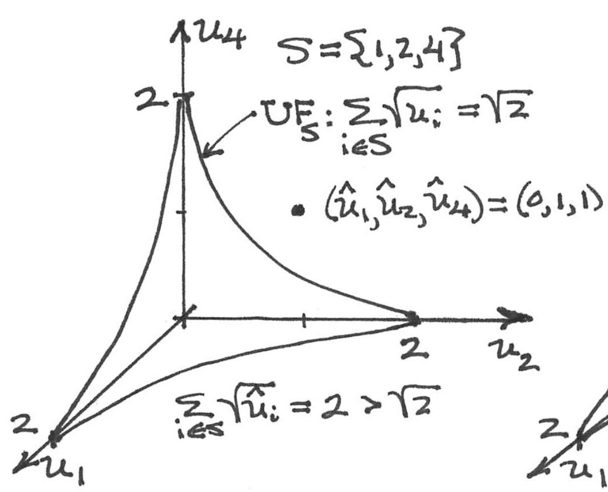


Figure 4

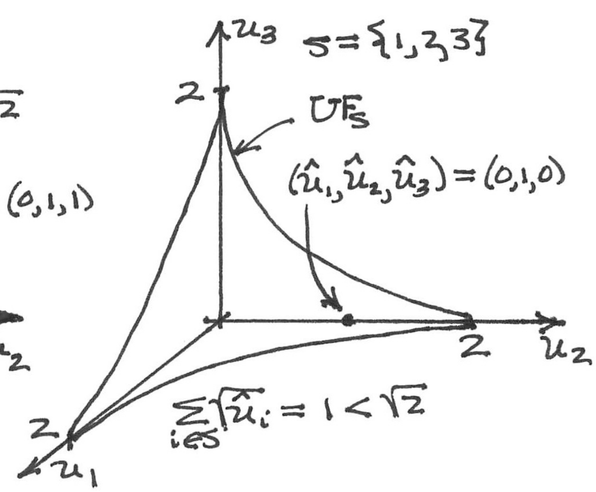


Figure 5

The three-person coalitions $S = \{1, 2, 3\}$ and $S = \{1, 3, 4\}$: (See Figure 5.)

Each of these coalitions has one even consumer and two odd consumers; we'll work with $S = \{1, 2, 3\}$. We have $(\mathring{x}_S, \mathring{y}_S) = (1, 2)$, so the utility frontier for S is given by the equation $\sqrt{u_1} + \sqrt{u_2} + \sqrt{u_3} = \sqrt{2}$. The utility profile $(\widehat{u}_1, \widehat{u}_2, \widehat{u}_3) = (0, 1, 0)$ is *inside* this utility frontier. The same is true for the coalition $S = \{1, 3, 4\}$, so either of these coalitions *can* unilaterally improve upon $(\widehat{x}^i, \widehat{y}^i)_N$.

To summarize, we've found two coalitions that can unilaterally improve upon the proposed allocation $(\widehat{x}^i, \widehat{y}^i)_N$, and therefore $(\widehat{x}^i, \widehat{y}^i)_N$ is not in the core of the four-person economy. While we haven't actually found a feasible allocation $(x^i, y^i)_S$ for a coalition S that's an improvement on $(\widehat{x}^i, \widehat{y}^i)_N$ for S, we nevertheless know that one exists, and that's enough to tell us that $(\widehat{x}^i, \widehat{y}^i)_N$ is not in the core.

Here's an allocation $(\widetilde{x}^i, \widetilde{y}^i)_S$ that's feasible for $S = \{1, 2, 3\}$ and that makes each consumer in S strictly better off than at $(\widehat{x}^i, \widehat{y}^i)_N$:

$$\left((\widetilde{x}^1, \widetilde{y}^1), (\widetilde{x}^2, \widetilde{y}^2), (\widetilde{x}^3, \widetilde{y}^3) \right) = \left((\tfrac{1}{8}, \tfrac{1}{4}), \ (\tfrac{3}{4}, \tfrac{3}{2}), \ (\tfrac{1}{8}, \tfrac{1}{4}) \right).$$

This allocation yields the utility profile $(\widetilde{u}^1, \widetilde{u}^2, \widetilde{u}^3) = (\frac{1}{32}, \frac{9}{8}, \frac{1}{32})$, which is larger in each component than $(\widehat{u}^1, \widehat{u}^2, \widehat{u}^3) = (0, 1, 0)$.

Notice that there are a lot of allocations to N *near* $(\widehat{x}^i, \widehat{y}^i)_N$ that are also unilaterally improved upon by $S = \{1, 2, 3\}$ with the S-allocation $\left((\widetilde{x}^1, \widetilde{y}^1), (\widetilde{x}^2, \widetilde{y}^2), (\widetilde{x}^3, \widetilde{y}^3) \right) = \left((\tfrac{1}{8}, \tfrac{1}{4}), \ (\tfrac{3}{4}, \tfrac{3}{2}), \ (\tfrac{1}{8}, \tfrac{1}{4}) \right)$. For example, consider the allocation

$$\left((x^1, y^1), (x^2, y^2), (x^3, y^3), (x^4, y^4) \right) = \left((\tfrac{1}{6}, \tfrac{1}{6}), \ (\tfrac{5}{6}, \tfrac{5}{6}), \ (\tfrac{1}{6}, \tfrac{1}{6}), (\tfrac{5}{6}, \tfrac{5}{6}) \right).$$

The S-allocation $(\widetilde{x}^i, \widetilde{y}^i)_S$ makes each consumer strictly better off. So *any* allocation to N that gives each of the odd consumers $\frac{1}{6}$ unit of each good, or less, is not in the core.

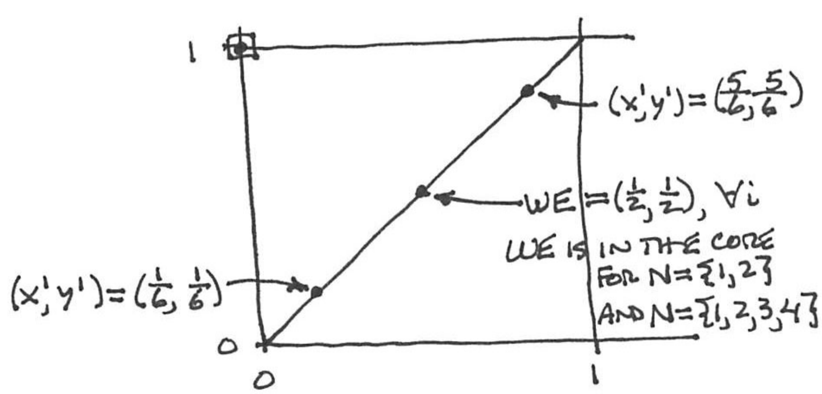


Figure 6

Exercise 1: Exactly which allocations of the form

$$((x^1, y^1), (x^2, y^2), (x^3, y^3), (x^4, y^4)) = ((a, a), (b, b), (a, a), (b, b))$$

are in the core of the four-consumer economy?

Exercise 2: Now assume there are r consumers of each type, with the utility function and initial bundles in the example. Determine which allocations of the form in Exercise 1 are core allocations. This requires two steps. The algebra in Step 1 is more difficult; but with Step 1 in hand, Step 2 is easy.

Step 1: Represent each coalition S by the pair (r, k), where r is the number of odd members and k is the number of even members. You need to show that the coalitions that can improve upon the most allocations are the ones in which $k = r - 1$.

Step 2: Determine for each r what is the smallest value of a for which the allocation in Exercise 1 can't be improved upon by a coalition with composition $(r, r - 1)$.

Notice how quickly the core becomes very small, if the only core allocations are the ones that have the form in Exercise 1. (And these are indeed the only core allocations.)

CORE ALLOCATIONS ARE NEARLY COMPETITIVE IF THE ECONOMY IS LARGE: FORMULATING THE IDEA

"IF THE ECONOMY IS LARGE, THEN A CORE ALLOCATION IS VERY NEARLY A COMPETITIVE ALLOCATION."

→ HOW LARGE? HOW NEAR?

ANALOGY:

"IF n IS A LARGE NUMBER, THEN $\frac{1}{n}$ IS VERY NEARLY ZERO."

DEF'N: IF THE NUMBER $a \in \mathbb{R}$ SATISFIES

$$\forall \varepsilon > 0 : \exists \hat{n} : n > \hat{n} \Rightarrow |x_n - a| < \varepsilon,$$

THEN WE SAY THAT a IS THE LIMIT OF THE SEQUENCE $\{x_n\}$, WHICH WE WRITE $\lim_{n \to \infty} x_n = a$.

"AS WE CONSIDER LARGER ECONOMIES, THE SET OF CORE ALLOCATIONS CONVERGES TO THE SET OF COMPETITIVE ALLOCATIONS."

DEF'NS:

(1) LET $B \subseteq \mathbb{R}^\ell$ AND $x \in \mathbb{R}^\ell$; $d(x,B) =_{def} \inf_{y \in B} \|x - y\|$.

(2) LET $A, B \subseteq \mathbb{R}^\ell$; $d(A,B) =_{def} \sup_{x \in A} d(x,B)$.

(3) $\lim_{n \to \infty} A_n = B$ IF $\lim_{n \to \infty} d(A_n, B) = 0$.

"IF $\{E(n)\}$ IS A SEQUENCE OF ECONOMIES IN WHICH, FOR EACH n, $E(n)$ HAS n CONSUMERS AND $W(n)$ AND $C(n)$ DENOTE THE SETS OF WALRAS AND CORE ALLOCATIONS, THEN $\lim_{n\to\infty} d(W(n) - C(n)) = 0$."

DIFFICULTIES:

① AS n CHANGES, $C(n)$ AND $W(n)$ LIE IN DIFFERENT SPACES: $C(n), W(n) \subseteq \mathbb{R}^{n\ell}$.

② CONCLUSION MAY BE UNTRUE IF $\{E(n)\}$ ITSELF IS NOT (IN SOME SENSE) CONVERGING. SUPPOSE, FOR EXAMPLE, THAT

$$\mathring{x}^i(n) = \begin{Bmatrix} (0,n) & \text{IF } i \text{ ODD} \\ (n,0) & \text{IF } i \text{ EVEN} \end{Bmatrix}; \quad u^i(n) = x_1^i x_2^i, \ \forall i, n.$$

"IF $\{E(n)\}$ CONVERGES AND $\lim_{n\to\infty} W(n) = W$, THEN $\lim_{n\to\infty} C(n) = W$ AS WELL."

... REQUIRES A DEF'N OF DISTANCE BETWEEN ALTERNATIVE PREFERENCES, AND WE AGAIN FACE DIFFICULTY ①.

IT IS BECOMING CLEAR THAT THIS WHOLE IDEA IS VERY DIFFICULT TO FORMULATE CLEARLY AND RIGOROUSLY — AND, ONCE THAT IS DONE, IT WILL PROBABLY BE DIFFICULT TO PROVE THE DESIRED THEOREM.

THE DEVELOPMENT OF THE IDEA:

EDGEWORTH (1881) [2x2 REPLICATED]

↓

DEBREU & SCARF (1963) [REPLICATION]

↓

AUMANN (1964) [CONTINUUM OF CONSUMERS]

↓

W. HILDENBRAND AND OTHERS (1966-1974)

[GENERAL RESULTS, USING CONTINUUM AS LIMIT OF LARGER AND LARGER FINITE ECONOMIES.]

LATER WORK (1975-1990): BROWN, KHAN, ANDERSON.

Large Economies: The Replication Model and Equal Treatment Allocations

Example 3 suggests that the core may be much smaller in a very large economy than it is when the economy has only a few consumers: with many consumers, there are more possibilities for coalitions to improve on proposed allocations. So perhaps, if the economy is large, the core will consist of only the Walrasian allocation(s)— which we know are in the core — and allocations extremely close to the Walrasian allocation(s).

Modeling this idea in a satisfactory way is not easy — for example, the dimension of the space of allocations depends on the number of consumers, so it's not clear how to represent the size of the set of core allocations for economies with different numbers of consumers. We tackle this problem with a very special way of modeling economies with different numbers of consumers: larger economies are modeled as replications of a smaller, basic economy.

The Basic Economy:

We start with a **basic economy** $E = (u^t, \mathring{\mathbf{x}}^t)_{t=1}^T$, which consists of T consumers. We refer to each of these consumers as one of the **types** that will be replicated. (Thus, the index t stands for **type**; and T is the set of all types as well as the number of types.) An allocation in E is, as usual, a T-tuple of bundles, $(\mathbf{x}^t)_T \in \mathbb{R}_+^{T\ell}$ — a bundle for each consumer in T.

The r-fold Replication of E:

In the **r-fold replication of** E, which we'll denote by $r * E$, there are r copies of each of the consumer types in the basic economy E. Thus, the economy $r * E$ has rT consumers, and we have to index them by indicating which type *and* which copy we're referring to:

$$\text{the } rT \text{ consumers in } r * E \text{ are indexed by } (t, q) \in r * T := T \times \{1, \ldots, r\}.$$

Equivalently, $r * T = \{(t, q) \mid t = 1, \ldots, T;\ q = 1, \ldots, r\}$. An allocation in $r * E$ is an rT-tuple of bundles, $(\mathbf{x}^{tq})_{r*T} \in \mathbb{R}_+^{rT\ell}$.

Allocations in $r * E$ that give the same bundle to every consumer of a given type will play a central role in the analysis. Because consumers of the same type get the same bundle in such an allocation, we refer to these allocations as *equal-treatment allocations*:

Definition: An **equal-treatment allocation** (abbreviated **ETA**) in $r * E$ is an allocation that satisfies the condition $\forall t, q, q' : \mathbf{x}^{tq} = \mathbf{x}^{tq'}$.

Example 3 was a replication economy in which $T = 2$ and $r = 2$. In $r * E$, all four consumers' utility functions were the same (*i.e.*, the two types, $t = 1$ and $t = 2$, happened to have the same utility function, although this would not be the case in general); and $\mathring{\mathbf{x}}^{11} = \mathring{\mathbf{x}}^{12} = (0, 1)$ and $\mathring{\mathbf{x}}^{21} = \mathring{\mathbf{x}}^{22} = (1, 0)$. In the example, we checked some ETAs (the ones that were replications of core allocations in the basic economy E) to determine which ones might be in the core of the replication economy $r * E$. We will show that *only* ETAs can be core allocations — *i.e.*, the core consists only of ETAs — so that the ETAs are the only allocations we need to check in order to determine whether they can be improved upon by some coalition. All other allocations *can* be improved upon.

First we have a useful remark and proposition:

Remark: If $(\mathbf{x}^{tq})_{r*T}$ is an ETA, then $\sum_{t=1}^{T}\sum_{q=1}^{r}\mathbf{x}^{tq} = \sum_{t=1}^{T} r\mathbf{x}^t$, where $\mathbf{x}^t = \mathbf{x}^{tq}, q = 1, \dots r$. In particular, $(\mathring{\mathbf{x}}^{tq})_{r*T}$, the initial allocation in $r * E$, is an ETA, and therefore any feasible allocation (even if it's *not* an ETA) must satisfy

$$\sum_{t=1}^{T}\sum_{q=1}^{r}\mathbf{x}^{tq} \leqq \sum_{t=1}^{T}\sum_{q=1}^{r}\mathring{\mathbf{x}}^{tq} = \sum_{t=1}^{T} r\mathring{\mathbf{x}}^t = r\sum_{t=1}^{T}\mathring{\mathbf{x}}^t.$$

Given any allocation $(\mathbf{x}^{tq})_{r*T}$ in $r * E$, let $\overline{\mathbf{x}}^t$ denote the mean bundle the consumers of type t receive in $(\mathbf{x}^{tq})_{r*T}$:

$$\overline{\mathbf{x}}^t := \frac{1}{r}\sum_{q=1}^{r}\mathbf{x}^{tq}.$$

The following proposition tells us that if the allocation $(\mathbf{x}^{tq})_{r*T}$ is feasible, then any coalition S consisting of exactly one consumer of each type can unilaterally obtain the allocation $(\overline{\mathbf{x}}^t)_T$:

Proposition: If $(\mathbf{x}^{tq})_{r*T}$ is feasible for $r * T$ in $r * E$, then $(\overline{\mathbf{x}}^t)_T$ is feasible for T.

Proof:

$$\begin{aligned}
\sum_{t=1}^{T}\overline{\mathbf{x}}^t &= \sum_{t=1}^{T}\frac{1}{r}\sum_{q=1}^{r}\mathbf{x}^{tq} \\
&= \frac{1}{r}\sum_{t=1}^{T}\sum_{q=1}^{r}\mathbf{x}^{tq} \\
&\leqq \frac{1}{r}\, r\sum_{t=1}^{T}\mathring{\mathbf{x}}^{tq}, \text{ by the Remark above, because } (\mathbf{x}^{tq})_{r*T} \text{ is feasible} \\
&= \sum_{t=1}^{T}\mathring{\mathbf{x}}^t. \qquad \|
\end{aligned}$$

Theorem: Let $E = (u^t, \mathring{\mathbf{x}}^t)_{t=1}^T$ be an economy in which each u^t is strictly quasiconcave, and let $r \in \mathbb{N}$. Then every core allocation in $r * E$ is an equal-treatment allocation.

Proof:

Let $(\mathbf{x}^{tq})_{r*T}$ be a feasible allocation for $r * E$. Wlog, suppose that for each type t, the first copy ($q = 1$) is treated the worst in $(\mathbf{x}^{tq})_{r*T}$ — *i.e.*,

$$\forall (t, q) \in r * T : \; u^t(\mathbf{x}^{t1}) \leqq u^t(\mathbf{x}^{tq}).$$

(Note that we haven't yet assumed that $(\mathbf{x}^{tq})_{r*T}$ is *not* an ETA: the inequality above could be satisfied weakly for all q.)

Let S be the coalition consisting of all the first-copy consumers: $S = \{(1, 1), (2, 1), \ldots, (T, 1)\}$. We have shown in the proposition above that the T-tuple of mean bundles $\overline{\mathbf{x}}^t$ is a feasible allocation for S. Moreover, for each type t, the mean bundle $\overline{\mathbf{x}}^t$ is a convex combination of the r bundles $\mathbf{x}^{t1}, \mathbf{x}^{t2}, \ldots, \mathbf{x}^{tr}$, all of which lie in the upper-contour set of $\mathbf{x}^{t1}$. Since the upper-contour set is convex, $\overline{\mathbf{x}}^t$ also lies in the upper-contour set — *i.e.*, $u^t(\overline{\mathbf{x}}^t) \geqq u^t(\mathbf{x}^{tq})$ for $q = 1, \ldots, r$.

Now suppose that $(\mathbf{x}^{tq})_{r*T}$ is not an ETA, and let t be any type for which all copies do not receive identical bundles — *i.e.*, there are some copies q and q' for which $\mathbf{x}^{tq} \neq \mathbf{x}^{tq'}$. Since all r coefficients are strictly positive when expressing $\overline{\mathbf{x}}^t$ as a convex combination of the bundles $\mathbf{x}^{tq}$ for $q = 1, \ldots, r$, and since u^t is strictly quasiconcave and all the bundles $\mathbf{x}^{tq}$ lie in the upper-contour set of $\mathbf{x}^{t1}$, we have $u^t(\overline{\mathbf{x}}^t) > u^t(\mathbf{x}^{t1})$. This establishes that the coalition S can unilaterally improve upon the non-ETA allocation $(\mathbf{x}^{tq})_{r*T}$. $\|$

Figures 1 and 2 depict the argument in the proof. In the figures, there are $r = 3$ copies of each type of consumer, and in each figure the copy $q = 1$ has the smallest utility level. In Figure 1 we have $u^1(\mathbf{x}^{11}) = u^1(\mathbf{x}^{12}) < u^1(\mathbf{x}^{13})$. In Figure 2 all three copies of type $t = 2$ have the same utility: copy $q = 1$ is no worse off than any other copy — his utility level is smallest, but it's not *smaller*.

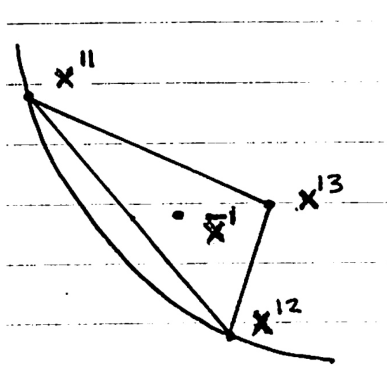


Figure 1

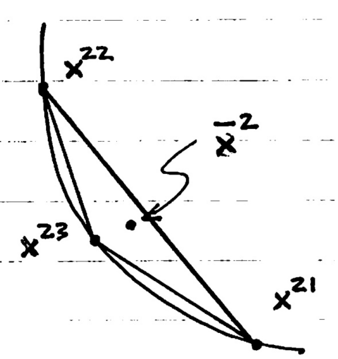


Figure 2

Figures 1 and 2 are intended only to show how the assumption of strictly convex preferences enters into the proof of the theorem. But it's patently clear on other grounds that the bundles in Figures 1 and 2 can't be part of a core allocation: the bundles assigned to various consumers of a given type have different MRS's and therefore can't be part of a Pareto allocation, and *a fortiori* they can't be part of a core allocation.

Here's an example where it's not at all obvious that the allocation can't be in the core. In order to show it, you need to essentially make the argument, for the example, that's in the proof of the theorem — or, of course, once we have the theorem, simply apply it to the allocation: since the allocation's not an ETA, it can't be in the core.

Example: There are two goods, two types, and two consumers of each type. The Type 1 consumers are both endowed with the bundle $(2,4)$ and the Type 2 consumers are both endowed with the bundle $(6,1)$; the economy's total endowment is therefore the bundle $(16,10)$. The proposed allocation is

$$\text{Type 1's:} \quad \mathbf{x}^{11} = (3,3), \quad \mathbf{x}^{12} = (6,1); \qquad \text{Type 2's:} \quad \mathbf{x}^{21} = (2,4), \quad \mathbf{x}^{22} = (5,2).$$

It's useful to see the same information presented in the following table:

	$\mathring{\mathbf{x}}^t$	$\mathbf{x}^{t1}$	$\mathbf{x}^{t2}$	$\overline{\mathbf{x}}^t$
$t=1$	$(2,4)$	$(3,3)$	$(6,1)$	$(4.5,2)$
$t=2$	$(6,1)$	$(2,4)$	$(5,2)$	$(3.5,3)$
$\Sigma_{t=1}^2$	$(8,5)$	$(5,7)$	$(11,3)$	$(8,5)$

Utility functions (or preferences) haven't been specified for the two types. Therefore it isn't clear, for either type, which of the two consumers of the type is worse off. You don't need to know which one is worse off to *apply* the theorem, but you would obviously have to know the preferences (or reproduce the argument in the proof) in order to verify directly that the proposed allocation isn't in the core.

The following exercise demonstrates why it's not obvious, without our theorem, that the proposed allocation isn't in the core (unlike in Figures 1 and 2, where the MRS's aren't all equal).

Exercise: Provide a geometric argument to show, in the above example, that there exist representable preferences (and therefore utility functions) in which the first consumer of each type is strictly worse off than the second consumer at the proposed allocation and all consumers have the same MRS (therefore the proposal is Pareto optimal — the four-player "coalition of the whole" can't improve on it). The same argument also shows that there exist preferences in which all consumers have the same MRS and the second consumer of each type is strictly worse off than the first consumer. **Hint:** In the first case it's clear that the common MRS must satisfy $MRS > 2/3$ and in the second case it's clear that we must have $MRS < 2/3$. It's also clear that in the $MRS > 2/3$ case the proposal can also be individually rational, so that none of the one-trader coalitions can improve on it either (by simply consuming his or her endowment bundle).

The Debreu-Scarf Theorem:
The Core Converges to the Walrasian Allocations

We've shown that any Walrasian equilibrium allocation (any WEA) is in the core, but it's obvious that the converse is far from true: most core allocations are not WEAs for the given initial distribution of goods. (Core allocations *are* Pareto efficient, so the Second Welfare Theorem does tell us that they can be supported as WEAs *if* we first implement some kind of redistribution.) But we saw, at least in an example, that some core allocations — the ones that are "farthest" from being Walrasian — were eliminated as we added consumers to the economy. The more consumers we added, the more allocations we eliminated: the additional consumers provided more opportunity to improve upon any proposed allocation. It seems reasonable to conjecture, then, that when the economy is very large (*i.e.*, when it has very many consumers), the core may consist only of WEAs and allocations very near them — *i.e.*, that core allocations are very nearly WEAs. And that perhaps "in the limit," core allocations *are* Walrasian equilibrium allocations.

As we've seen, merely stating this idea formally is difficult. We'll take the approach that Edgeworth took when he first came up with this idea, and which Debreu and Scarf finally formalized and used to prove the conjecture many decades later — the idea of considering ever-larger replications of a basic economy. In this framework, the theorem we state and prove (for the 2×2 case) says that for any allocation that's not a WEA, if we make the economy large enough (*i.e.*, if we replicate it sufficiently many times), it will be so large that the non-WEA we started with will fail to be in the large economy's core.

Theorem: Let $E = (u^t, \mathring{\mathbf{x}}^t)_{t=1}^T$ be an economy in which each u^t is continuous, strictly quasiconcave, and strictly increasing, and in which $\mathring{\mathbf{x}}^t_k > 0$ for each $t \in T$ and each good $k = 1, \ldots, \ell$. If an allocation $(\mathbf{x}^t)_T \in \mathbb{R}^{T\ell}_+$ is not a Walrasian equilibrium allocation for E, then there is an integer $\widehat{r}$ such that, for all $r \geqq \widehat{r}$, the allocation $r * (\mathbf{x}^t)_T$ is not in the core of the replication economy $r * E$.

Proof: (For the 2×2 case — 2 persons, 2 goods)

(This proof assumes that each u^t is differentiable. This is not essential, but it makes the proof more transparent.)

Suppose that $((\widehat{x}^1, \widehat{y}^1), (\widehat{x}^2, \widehat{y}^2))$, or $(\widehat{x}^t, \widehat{y}^t)_T$ for short, is in the core, but is not a Walrasian equilibrium allocation (a WEA). We will show that if r is large enough, then the r-fold replication of $(\widehat{x}^t, \widehat{y}^t)_T$ — *i.e.*, $r * (\widehat{x}^t, \widehat{y}^t)_T$ — will *not* be in the core of the r-fold replication $r * E$.

First, notice that $(\widehat{x}^t, \widehat{y}^t)_T \neq (\mathring{x}^t, \mathring{y}^t)_T$: we've assumed that $(\widehat{x}^t, \widehat{y}^t)_T$ is in the core, so it is Pareto efficient; but if the endowment allocation is Pareto efficient, then the Second Welfare Theorem would ensure that it's a WEA, and we've assumed that $(\widehat{x}^t, \widehat{y}^t)_T$ is *not* a WEA.

Let L denote the line that passes through, say, $(\widehat{x}^1, \widehat{y}^1)$ and $(\mathring{x}^1, \mathring{y}^1)$, and let $-\tau$ be its slope:

$$\tau = -\frac{\widehat{y}^1 - \mathring{y}^1}{\widehat{x}^1 - \mathring{x}^1} = -\frac{\widehat{y}^2 - \mathring{y}^2}{\widehat{x}^2 - \mathring{x}^2},$$

which is the trading ratio defined by $(\widehat{x}^t, \widehat{y}^t)_T$ and $(\mathring{x}^t, \mathring{y}^t)_T$. [The two traders' trading ratios are equal because $\widehat{x}^1 + \widehat{x}^2 = \mathring{x}$ and $\widehat{y}^1 + \widehat{y}^2 = \mathring{y}$, which follows from the fact that each u^t is increasing.] Wlog, assume that the common MRS at $(\widehat{x}^t, \widehat{y}^t)_T$, denoted σ, satisfies $\sigma < \tau$, and assume that $\widehat{x}^1 > \mathring{x}^1$ and $\widehat{x}^2 < \mathring{x}^2$. [The common MRS exists because $(\widehat{x}^t, \widehat{y}^t)_T$ is Pareto efficient and preferences are quasiconcave.]

Since $\sigma < \tau$, each consumer would gain by giving up some (perhaps only very little) of the x-good in return for the y-good *at the rate* τ, as depicted in Figure 1. If we write

$$\mathbf{z}^t = (z_x^t, z_y^t) = (x^t - \mathring{x}^t,\ y^t - \mathring{y}^t)$$

for type t's net trades, and

$$\widetilde{u}^t(\mathbf{z}^t) := u^t(\mathring{x}^t + z_x^t,\ \mathring{y}^t + z_y^t)$$

for type t's utility from a net trade $\mathbf{z}^t$, then we have

(1) $\qquad \widehat{\mathbf{z}}^1 + \widehat{\mathbf{z}}^2 = (0,\ 0)$, and

(2) $\qquad \widetilde{u}^1(\lambda_1 \widehat{\mathbf{z}}^1) > \widetilde{u}^1(\widehat{\mathbf{z}}^1)$ and $\widetilde{u}^2(\lambda_2 \widehat{\mathbf{z}}^2) > \widetilde{u}^2(\widehat{\mathbf{z}}^2)$

for some $\lambda_1 < 1$ and $\lambda_2 > 1$. We need to construct a coalition made up of α_1 members of type 1 and α_2 members of type 2, and give each member of the coalition the net trade $\lambda_t \widehat{\mathbf{z}}^t$ (depending on the member's type, $t = 1$ or $t = 2$), thereby making each member better off than at $\widehat{\mathbf{z}}^t$. The question is: How can we use the numbers $\lambda_1 < 1$ and $\lambda_2 > 1$ to determine the numbers α_1 and α_2 (which must be integers)?

If each member of the coalition receives the net trade $\widehat{\mathbf{z}}^t$ $(t = 1, 2)$, then the coalition's aggregate net trade will be $\alpha_1 \lambda_1 \widehat{\mathbf{z}}^1 + \alpha_2 \lambda_2 \widehat{\mathbf{z}}^2$. That aggregate net trade has to be $(0, 0)$ if the coalition is to implement it unilaterally. Therefore we need to have

$$\alpha_1 \lambda_1 \widehat{\mathbf{z}}^1 + \alpha_2 \lambda_2 \widehat{\mathbf{z}}^2 = (0, 0).$$

Since we do have $\widehat{\mathbf{z}}^1 + \widehat{\mathbf{z}}^2 = (0,\ 0)$, it will suffice to have

(3) $\qquad \alpha_1 \lambda_1 = \alpha_2 \lambda_2; \quad i.e., \ \dfrac{\alpha_2}{\alpha_1} = \dfrac{\lambda_1}{\lambda_2}.$

If λ_1 and λ_2 are rational numbers (each a ratio of integers), then we can choose integers α_1 and α_2 that satisfy (3), and then we let $\widehat{r} = \max\{\alpha_1,\ \alpha_2\}$. And it's clear from (2), together with continuity of each u^t, that we can indeed choose λ_1 and λ_2 to be rational. □

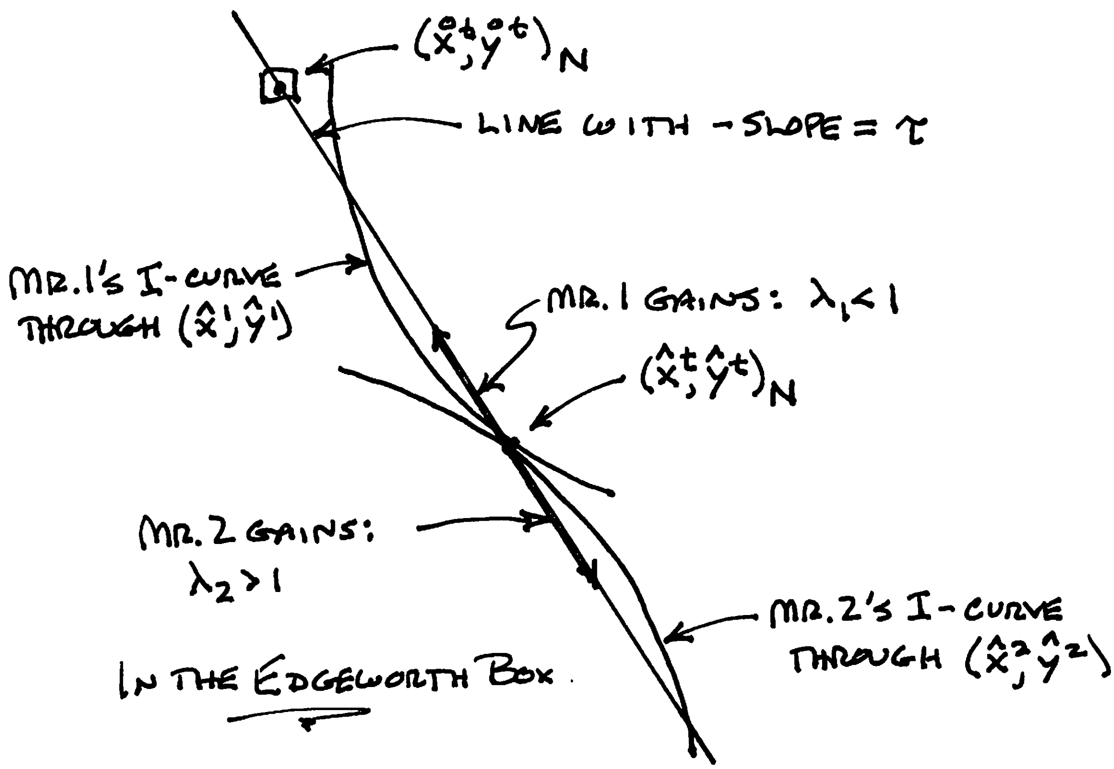


FIGURE 1

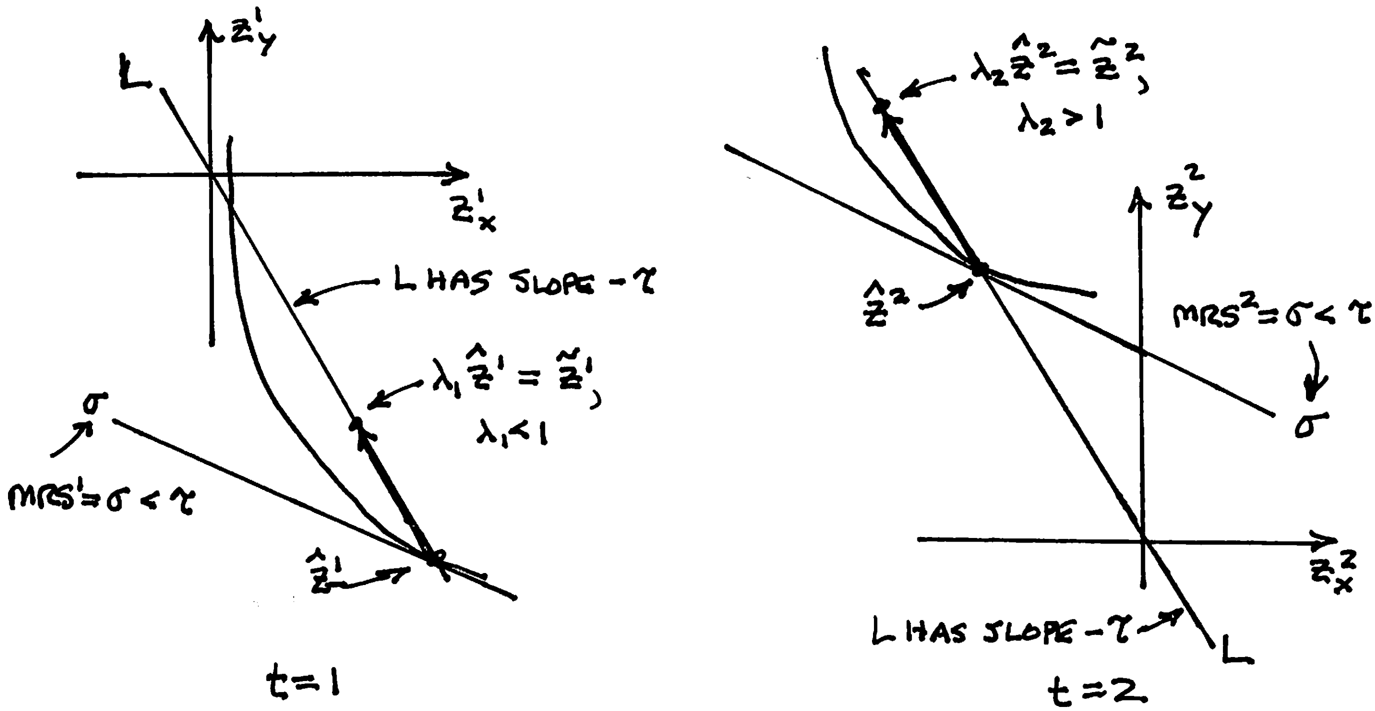


FIGURE 2

THE PIECES OF THE ARGUMENT

(1) $\tau := -\frac{\hat{z}_{1y}}{\hat{z}_{1x}} = -\frac{\hat{z}_{2y}}{\hat{z}_{2x}}$ OK, BECAUSE (a) $\hat{z}_{1x}, \hat{z}_{2x} \neq 0$ [IF THE ENDOWMENT IS IN THE CORE, IT'S COMPETITIVE], AND (b) $((\hat{x}_i, \hat{y}_i))_N$ USES ALL OF $(\mathring{x}, \mathring{y})$, BECAUSE EACH u_i INCREASING.

(2) $MRS_1 = MRS_2 = \sigma$, SAY. BECAUSE $((\hat{x}_i, \hat{y}_i))_N$ IN THE CORE, AND EACH u_i QUASI-CONCAVE.

(3) $\tau \neq \sigma$. OTHERWISE $((\hat{x}_i, \hat{y}_i))_N$ IS COMPETITIVE.

(4) WLOG: $\sigma < \tau$; $\hat{x}_1 > \mathring{x}_1$;
∴ $\hat{x}_2 < \mathring{x}_2$; $\hat{y}_1 < \mathring{y}_1$, $\hat{y}_2 > \mathring{y}_2$.

(5) $\exists\ \lambda_1 < 1$ AND $\lambda_2 > 1$ s.t.
$\tilde{u}_1(\lambda_1 \hat{z}_1) > \tilde{u}_1(\hat{z}_1)$ AND $\tilde{u}_2(\lambda_2 \hat{z}_2) > \tilde{u}_2(\hat{z}_2)$.

(6) IF A COALITION WITH α_i OF TYPE i $(i=1,2)$ GIVES $\lambda_i \hat{z}_i$ TO EACH MEMBER, ALL ARE BETTER OFF THAN AT $\hat{z}_i$. THIS WILL BE FEASIBLE FOR THE COALITION IF $\alpha_1 \lambda_1 \hat{z}_1 + \alpha_2 \lambda_2 \hat{z}_2 = (0,0)$; i.e., IF $\alpha_1 \lambda_1 = \alpha_2 \lambda_2$ [SINCE $\hat{z}_1 + \hat{z}_2 = (0,0)$].

(7) WE WANT INTEGERS α_1, α_2 s.t. $\frac{\alpha_2}{\alpha_1} = \frac{\lambda_1}{\lambda_2}$, WHICH WE CAN DO IF λ_1/λ_2 IS RATIONAL. CONTINUITY OF EACH u_i ENSURES THERE WILL BE SUCH λ_i's.

EXAMPLE:

SUPPOSE $\lambda_1 = \frac{8}{9}$ AND $\lambda_2 = \frac{6}{5}$.

THEN $\frac{\lambda_1}{\lambda_2} = \frac{8/9}{6/5} = \frac{40}{54} = \frac{20}{27}$.

SO LET $\alpha_1 = 27$ AND $\alpha_2 = 20$, WHICH REQUIRES $r \geq 27$. THIS YIELDS

$$\begin{aligned} \alpha_1 \lambda_1 \hat{z}_1 + \alpha_2 \lambda_2 \hat{z}_2 &= 24\hat{z}_1 + 24\hat{z}_2 \\ &= 24(\hat{z}_1 + \hat{z}_2) \\ &= 24(0,0) \\ &= (0,0), \end{aligned}$$

SO GIVING $\lambda_1 \hat{z}_1$ TO EACH TYPE 1 AND $\lambda_2 \hat{z}_2$ TO EACH TYPE 2

IS UNILATERALLY FEASIBLE FOR A COALITION WITH $\alpha_1 = 27$ TYPE 1'S AND $\alpha_2 = 20$ TYPE 2'S.

CONSEQUENTLY, IF $r \geq 27$ THEN THERE IS A COALITION THAT CAN UNILATERALLY IMPROVE UPON AN ALLOCATION GIVING NET TRADES $\hat{z}_1$ AND $\hat{z}_2$. (ASSUMING λ_1 AND λ_2 SATISFY $\tilde{u}_i(\lambda_i \hat{z}_i) > \tilde{u}_i(\hat{z}_i)$, $i = 1, 2$.)

Another Example:

Suppose $\lambda_1 = \frac{8}{9}$, $\lambda_2 = \sqrt{2} \cong 1.414$

If we want $\frac{\alpha_2}{\alpha_1} = \frac{\lambda_1}{\lambda_2}$, we have $\alpha_2 = \frac{8}{9\sqrt{2}} \alpha_1$,

i.e., $\alpha_1 = \frac{9}{8}\sqrt{2}\,\alpha_2$, and there ~~are~~ are no <u>integers</u> α_1 and α_2 that satisfy this equation.

[If α_2 is an integer, α_1 is clearly not.]

But there has to be some $\tilde{\lambda}_2$ <u>near</u> λ_2 satisfying

(1) $\tilde{\lambda}_2$ is rational (a ratio of integers),

and (2) type 2's prefer the trade $\tilde{\lambda}_2 \hat{z}_2$ to the proposal, $\hat{z}_2$.

[This $\tilde{\lambda}_2$ may have to be chosen <u>very</u> near λ_2.]

If, for example, $\tilde{\lambda}_2 = 1.4 = \frac{7}{5}$ yields a $\tilde{\lambda}_2 \hat{z}_2$ preferred to $\hat{z}_2$, then we can use

$$\frac{\alpha_2}{\alpha_1} = \frac{8/9}{7/5} = \frac{40}{63} : \quad r \geq 63.$$

But if 1.4 <u>doesn't</u> satisfy $\tilde{\lambda}_2 \hat{z}_2 \succ \hat{z}_2$, perhaps $\tilde{\lambda}_2 = 1.41$ does, etc.

Concluding Remarks:

The way we've modeled large economies is extremely special and unrealistic. An actual economy, if it's very large, isn't going to consist of only a small number of types of consumer, with every consumer being one of these few types. Even if this were a good approximation — even if there were a small number of types and every consumer were very close to one of those types — it would be astonishing if there were also *exactly the same number* of consumers of each type.

Shortly after Debreu and Scarf published their paper on the core convergence theorem, in 1963, Robert Aumann published a paper in which he took a remarkably innovative approach to formulating a model of a large economy in which individual consumers have negligible influence. Aumann modeled a large economy as one with an infinite set of consumers, endowed with a measure in which each individual consumer has measure zero. Within this model, Aumann used essentially the Debreu-Scarf method of proof to show that in a large economy the only core allocations are the Walrasian equilibrium allocations. Aumann's paper — especially the introductory section — is one of the most striking and elegant papers in economics. You should definitely read both the introductory and concluding sections, and make the effort to read the remaining five pages which contain the formal model and proof. The paper is available on the course website, in the Readings section.

After Aumann's paper, a great deal of work was devoted to these ideas over the subsequent two or three decades, in which Aumann's continuum model (and the core equivalence result, and others) was shown to be the limiting case, in a well-defined sense, of large but finite economies.

So what's the significance of the Core Convergence (Debreu-Scarf) Theorem? It tells us that if the economy is sufficiently large that individual consumers are negligible, then whatever institution we use to allocate resources, we will end up with the same outcome we would have attained via markets and prices. Of course, that assumes we have no externalities, consumers have complete information about the prices and the commodities, and consumers are free to "go their own way," using their own resources independently of other consumers. And note that we didn't allow production, which complicates things considerably, largely because of scale phenomena.

The concept of the core is important in contexts other than large economies. For one example, in auction theory, see the paper by Ausubel and Milgrom on the course website, especially Section 5 of the paper.

Exercise: The Core Shrinks Under Replication

We begin with a 2×2 "Edgeworth Box" exchange economy: each consumer has the same preference, described by the utility function $u(x, y) = xy$; Consumer 1 owns the bundle $(\mathring{x}_1, \mathring{y}_1) = (15, 30)$; and Consumer 2 owns the bundle $(\mathring{x}_2, \mathring{y}_2) = (75, 30)$.

(a) Verify that there is a unique Walrasian (competitive) equilibrium, in which the price ratio is $p_x/p_y = 2/3$ and the consumption bundles are $(x_1, y_1) = (30, 20)$ and $(x_2, y_2) = (60, 40)$.

(b) Verify that the Pareto allocations are the ones that allocate the entire resource endowment of $(\mathring{x}, \mathring{y}) = (90, 60)$ and satisfy $y_1/x_1 = y_2/x_2 = 2/3$.

(c) In the Edgeworth Box draw the competitive allocation, the Pareto allocations, and each consumer's budget constraint at the competitive prices. Draw each consumer's indifference curve containing his initial bundle and indicate the core allocations in the diagram.

(d) Verify that the Pareto allocations for which $x_1 < \sqrt{675}$ are not in the core. Note that $\sqrt{675}$ is approximately 26. Similarly, the Pareto allocations for which $x_2 < \sqrt{3375} \approx 58.1$ are not in the core.

(e) Consider a proposed allocation $(\hat{x}_1, \hat{y}_1) = (27, 18)$ and $(\hat{x}_2, \hat{y}_2) = (63, 42)$. Note that each consumer's marginal rate of substitution at the proposal is 2/3. Verify that the proposal is in the core. Verify that the "trading ratio" τ defined by the proposal is $\tau = 1$. As in our lecture notes on the Debreu-Scarf Theorem, use the "shrinkage factor" $\lambda_1 = 2/3$ and the "expansion factor" $\lambda_2 = 4/3$ to verify that a coalition of just two "Type 1" consumers and one "Type 2" consumer can unilaterally allocate their initial bundles to make all three of them better off than in the proposal. Therefore the proposal is not in the core if there are two or more consumers of each type.

(f) Now consider the proposal $(\hat{x}_1, \hat{y}_1) = (28\frac{1}{2}, 19)$ and $(\hat{x}_2, \hat{y}_2) = (61\frac{1}{2}, 41)$, and use the same λ_1 and λ_2 as in (e) to establish that this proposal too is not in the core if there are two or more consumers of each type.

(g) Now consider the proposal $(\hat{x}_1, \hat{y}_1) = (29, 19\frac{1}{3})$ and $(\hat{x}_2, \hat{y}_2) = (61, 40\frac{2}{3})$, and use the factors $\lambda_1 = 4/5$ and $\lambda_2 = 6/5$ to establish that this proposal is not in the core if there are three or more consumers of each type.