A Core Example with Four Consumers

In this example we're going to have four consumers, indexed by $i \in N = \{1, 2, 3, 4\}$. The two "odd" consumers ($i = 1, 3$) will be identical and the two "even" consumers ($i = 2, 4$) will be identical. The example will show how it is that adding more consumers to the economy can remove some allocations from the core, in a certain sense, thus shrinking the core to a smaller set of allocations.

Each consumer in the example has the same utility function, $u(x, y) = xy$. The two odd consumers are each endowed with one unit of the y-good and none of the x-good, and each of the even consumers is endowed with one unit of the x-good and none of the y-good:

$$(\mathring{x}^1, \mathring{y}^1) = (\mathring{x}^3, \mathring{y}^3) = (0, 1) \qquad \text{and} \qquad (\mathring{x}^2, \mathring{y}^2) = (\mathring{x}^4, \mathring{y}^4) = (1, 0).$$

The allocations for this economy are in $\mathbb{R}^8_+$, so we're not going to get very far trying to use an Edgeworth box-type diagram, which will be six-dimensional. But we can exploit the fact that there are only two distinct "types" of consumer, the odds and the evens, by first analyzing an economy in which there is only *one* consumer of each type, say just $i = 1$ and $i = 2$. In this two-person economy it's easy to see that the Pareto allocations are the ones in which $x^1 = y^1$ and $x^2 = y^2$ — *i.e.*, the diagonal of the Edgeworth box (EB). The unique Walrasian equilibrium (WE) has $p_x = p_y$ and each consumer receives the bundle $(x^i, y^i) = (\frac{1}{2}, \frac{1}{2})$. See Figure 1.

Now returning to the four-person economy, the Walrasian equilibrium still has $p_x = p_y$, and each consumer still receives the bundle $(x^i, y^i) = (\frac{1}{2}, \frac{1}{2})$. The Pareto allocations are still the ones in which $x^i = y^i$ for each consumer. So the Edgeworth box is still somewhat useful here: in the WE, each consumer of a given type receives the bundle he would receive in the Edgeworth box economy, and the Pareto set is similar to the Pareto set in the EB economy.

What about the core in the two economies? In the two-person economy, with one consumer of each type, the core is the entire diagonal of the box. In particular, the lower-left-corner allocation, in which $(x^1, y^1) = (0, 0)$ and $(x^2, y^2) = (1, 1)$, is in the core. In the corresponding four-person allocation, both odd consumers receive $(0, 0)$ and both even consumers receive $(1, 1)$. We'll denote this allocation by $(\widehat{x}^i, \widehat{y}^i)_N$, where $N = \{1, 2, 3, 4\}$. See Figure 1. Is $(\widehat{x}^i, \widehat{y}^i)_N$ in the core of the four-person economy? Let's figure it out.

(The corner allocation, in which the odd consumers each get $(0, 0)$, may seem kind of extreme. Before we're done we'll see that the argument we're going to develop applies to plenty of not-so-extreme allocations too. In fact, to all but the WE allocation, where each $(x^i, y^i) = (\frac{1}{2}, \frac{1}{2})$!)

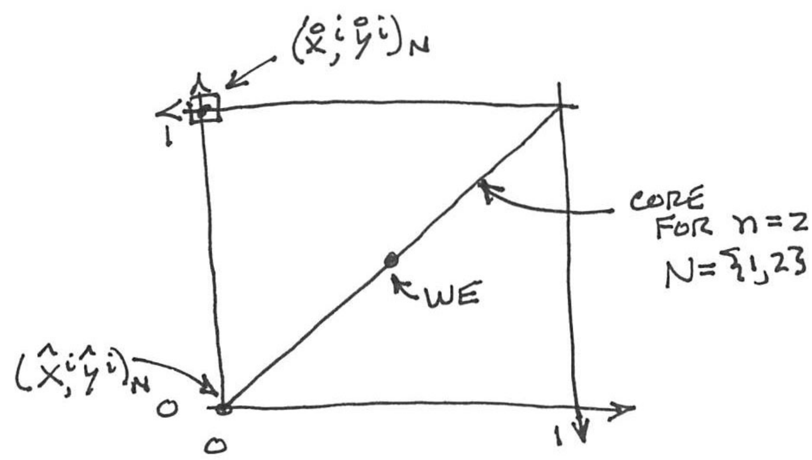

Figure 1

How can we show that a given allocation, such as $(\widehat{x}^i, \widehat{y}^i)_N$, is in the core, or alternatively that it's *not* in the core? If we can find just one coalition S that can unilaterally improve on $(\widehat{x}^i, \widehat{y}^i)_N$, then it's not in the core. Conversely, in order to show that the allocation *is* in the core, we have to show that none of the fifteen possible coalitions can unilaterally improve on it.

It's clear that the four-person "coalition of the whole," $S = N$, can't improve on $(\widehat{x}^i, \widehat{y}^i)_N$, because $(\widehat{x}^i, \widehat{y}^i)_N$ is Pareto efficient. It's also clear that none of the one-person coalitions $\{1\}, \{2\}, \{3\}$, or $\{4\}$ can improve, because each consumer is receiving at least as much utility in $(\widehat{x}^i, \widehat{y}^i)_N$ as she receives by just consuming her initial bundle.

For most of the remaining coalitions (*i.e.*, the two- and three-person coalitions), it's also not too difficult to show that the coalition can't improve on $(\widehat{x}^i, \widehat{y}^i)_N$. For example, the coalition $S = \{1, 3\}$ consisting of just the two odd consumers owns two units of the y-good but none of the x-good, so even though each of these two consumers receives $u_i = 0$ in $(\widehat{x}^i, \widehat{y}^i)_N$, they can't do any better than that with just their own resources.

However, there's a more systematic way to determine whether coalitions can improve on a given allocation such as $(\widehat{x}^i, \widehat{y}^i)_N$: we can use the coalition's utility frontier. Recall from our Utility Frontier notes that we used the notation $\mathcal{F}$ to denote the set of feasible allocations for the set N of all the consumers — *i.e.*, the allocations that satisfy $\sum_{i \in N} \mathbf{x}^i \leqq \mathring{\mathbf{x}}$. We used $\mathcal{P}$ to denote the set of Pareto allocations, and we used U to denote the function that maps allocations $(\mathbf{x}^i)_N$ to their resulting utility profiles $(u_i)_N = U((\mathbf{x}^i)_N) = (u^1(\mathbf{x}^1), \ldots, u^n(\mathbf{x}^n))$. In the space $\mathbb{R}^n$ of utility profiles, $U(\mathcal{F})$ is the set of feasible utility profiles — the image under U of the set of feasible allocations — and $U(\mathcal{P})$ is the utility frontier, the image under U of the set of Pareto efficient allocations.

Let's replace the set N everywhere in the preceding paragraph with a coalition $S \subseteq N$, and write $\mathcal{F}_S$ for the set of allocations $(x^i)_S$ to S that are feasible for S — *i.e.*, $\sum_{i\in S} \mathbf{x}^i \leqq \mathring{\mathbf{x}}_S$ — and write $\mathcal{P}_S$ for the set of allocations to S that are Pareto efficient for S using just its own resources. Now consider an allocation $(\widehat{\mathbf{x}}^i)_N$ to N and the associated utility profile $(\widehat{u}_i)_N$. A coalition S can unilaterally improve upon $(\widehat{\mathbf{x}}^i)_N$ exactly if there is some allocation $(\mathbf{x}^i)_S$ to S that's feasible for S and a Pareto improvement (for S!) upon $(\widehat{\mathbf{x}}_i)_S$ — in other words, S can unilaterally improve if S's part of the profile $(\widehat{u}_i)_N$, namely $(\widehat{u}_i)_S$, is in $U(\mathcal{F}_S)$ but not on the utility frontier, $U(\mathcal{P}_S)$. If the utility frontier for S is described by the equation $h_S((u_i)_S) = c$ for some real number c and some strictly increasing function h_S, then S can unilaterally improve upon $(\widehat{\mathbf{x}}^i)_N$ if and only if

$$h_S((\widehat{u}_i)_S) < c\,, \tag{1}$$

In short, S can unilaterally improve if and only if $(\widehat{u}_i)_S$ lies strictly inside S's utility frontier.

Let's see how this works in our four-consumer example. In the Utility Frontier notes we saw that if a set S of consumers has a total of $\mathring{x}_S$ units of the x-good and $\mathring{y}_S$ units of the y-good, and if every one of the consumers has the utility function $u(x,y) = xy$, then the utility frontier is the set of utility profiles that satisfy the equation

$$\sum_{i\in S} \sqrt{u_i} = \sqrt{\mathring{x}_S \mathring{y}_S}\,. \tag{2}$$

Since the coalition of the whole, $S = N$, has $(\mathring{x}, \mathring{y}) = (2,2)$, the utility frontier for N is $\sum_{i\in N} \sqrt{u_i} = 2$. The proposed allocation $(\widehat{x}^i, \widehat{y}^i)_N$, in which each odd consumer receives $(0,0)$ and each even consumer receives $(1,1)$, yields the utility profile $(\widehat{u}_i)_{i\in N} = (0,1,0,1)$, which is *on* the utility frontier for N — which we knew it must be, because $(\widehat{x}^i, \widehat{y}^i)_N$ is Pareto efficient.

Let's look at all the other coalitions:

The one-person coalitions, $S = \{1\}, \{2\}, \{3\}, \{4\}$:

UF$_S$ is $u_i = u(\mathring{x}^i, \mathring{y}^i) = 0$. Since $\widehat{u}_i = 0$ for i odd and $\widehat{u}_i = 1$ for i even, $\widehat{u}_i$ is either on or outside the utility frontier for each i. So none of these coalitions can unilaterally improve upon $(\widehat{x}^i, \widehat{y}^i)_N$.

The two-person coalitions $S = \{1,2\}, \{1,4\}, \{3,2\}, \{3,4\}$: (See Figure 2.)

Each of these is just like the two-person Edgeworth box economy, with one odd consumer and one even, and UF$_S$ is given by $\sqrt{u_{odd}} + \sqrt{u_{even}} = 1$. Since $\widehat{u}_{odd} = 0$ and $\widehat{u}_{even} = 1$, we have $(\widehat{u}_{odd}, \widehat{u}_{even}) = (0,1)$, which is on the utility frontier for S. So none of these coalitions can unilaterally improve upon $(\widehat{x}^i, \widehat{y}^i)_N$.

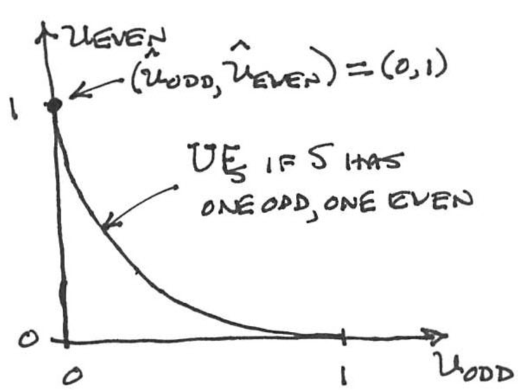

Figure 2

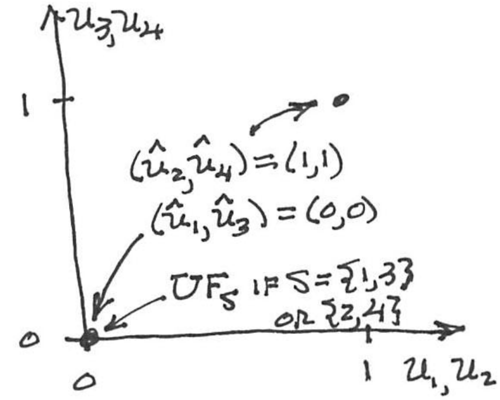

Figure 3

The two-person coalitions $S = \{1,3\}$ and $S = \{2,4\}$: (See Figure 3.)

In each of these two cases the coalition has none of one of the two goods, so its utility frontier is given by $\sum_{i \in S} \sqrt{u_i} = 0$, and the utility frontier is therefore the singleton $\{(0,0)\}$. For $S = \{1,3\}$, the utility profile $(\widehat{u}_1, \widehat{u}_3)$ is $(0,0)$, which is on the utility frontier. For $S = \{2,4\}$, the utility profile $(\widehat{u}_2, \widehat{u}_4)$ is $(1,1)$, which is outside the utility frontier. So neither of these coalitions can unilaterally improve upon $(\widehat{x}^i, \widehat{y}^i)_N$.

The three-person coalitions $S = \{1,2,4\}$ and $S = \{3,2,4\}$: (See Figure 4.)

Each of these coalitions has one odd consumer and two even consumers; we'll work with $S = \{1,2,4\}$. We have $(\mathring{x}_S, \mathring{y}_S) = (2,1)$, so the utility frontier for S is given by the equation $\sqrt{u_1} + \sqrt{u_2} + \sqrt{u_4} = \sqrt{2}$. The utility profile $(\widehat{u}_1, \widehat{u}_2, \widehat{u}_4) = (0,1,1)$ is outside this utility frontier. The same is true for $S = \{3,2,4\}$, so neither of these coalitions can unilaterally improve upon $(\widehat{x}^i, \widehat{y}^i)_N$.

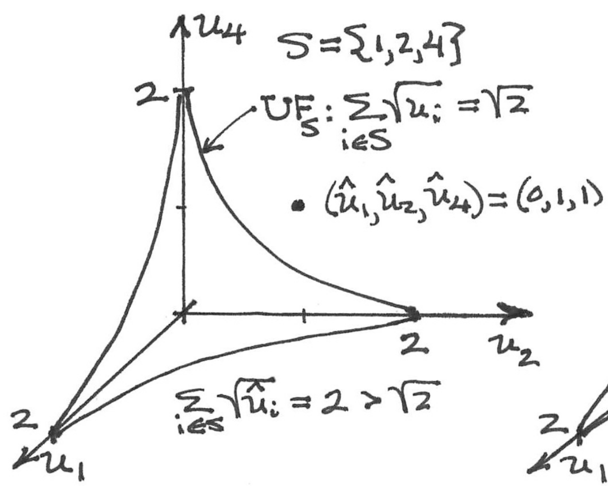

Figure 4

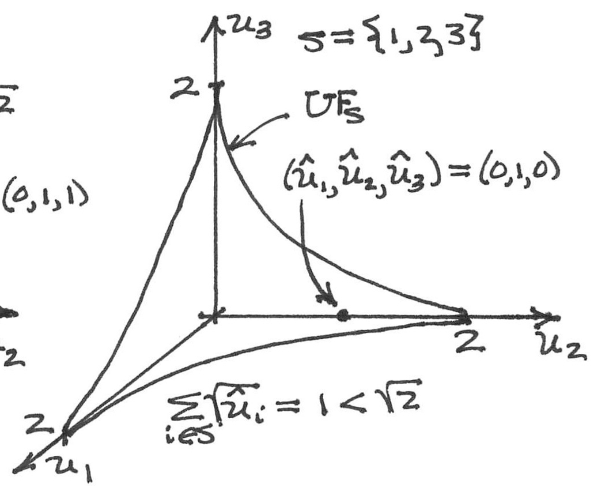

Figure 5

The three-person coalitions $S = \{1,2,3\}$ and $S = \{1,3,4\}$: (See Figure 5.)

Each of these coalitions has one even consumer and two odd consumers; we'll work with $S = \{1,2,3\}$. We have $(\mathring{x}_S, \mathring{y}_S) = (1,2)$, so the utility frontier for S is given by the equation $\sqrt{u_1} + \sqrt{u_2} + \sqrt{u_3} = \sqrt{2}$. The utility profile $(\widehat{u}_1, \widehat{u}_2, \widehat{u}_3) = (0,1,0)$ is *inside* this utility frontier. The same is true for the coalition $S = \{1,3,4\}$, so either of these coalitions *can* unilaterally improve upon $(\widehat{x}^i, \widehat{y}^i)_N$.

To summarize, we've found two coalitions that can unilaterally improve upon the proposed allocation $(\widehat{x}^i, \widehat{y}^i)_N$, and therefore $(\widehat{x}^i, \widehat{y}^i)_N$ is not in the core of the four-person economy. While we haven't actually found a feasible allocation $(x^i, y^i)_S$ for a coalition S that's an improvement on $(\widehat{x}^i, \widehat{y}^i)_N$ for S, we nevertheless know that one exists, and that's enough to tell us that $(\widehat{x}^i, \widehat{y}^i)_N$ is not in the core.

Here's an allocation $(\widetilde{x}^i, \widetilde{y}^i)_S$ that's feasible for $S = \{1,2,3\}$ and that makes each consumer in S strictly better off than at $(\widehat{x}^i, \widehat{y}^i)_N$:

$$\left((\widetilde{x}^1, \widetilde{y}^1), (\widetilde{x}^2, \widetilde{y}^2), (\widetilde{x}^3, \widetilde{y}^3) \right) = \left((\tfrac{1}{8}, \tfrac{1}{4}),\ (\tfrac{3}{4}, \tfrac{3}{2}),\ (\tfrac{1}{8}, \tfrac{1}{4}) \right).$$

This allocation yields the utility profile $(\widetilde{u}^1, \widetilde{u}^2, \widetilde{u}^3) = (\frac{1}{32}, \frac{9}{8}, \frac{1}{32})$, which is larger in each component than $(\widehat{u}^1, \widehat{u}^2, \widehat{u}^3) = (0,1,0)$.

Notice that there are a lot of allocations to N *near* $(\widehat{x}^i, \widehat{y}^i)_N$ that are also unilaterally improved upon by $S = \{1,2,3\}$ with the S-allocation $\left((\widetilde{x}^1, \widetilde{y}^1), (\widetilde{x}^2, \widetilde{y}^2), (\widetilde{x}^3, \widetilde{y}^3) \right) = \left((\tfrac{1}{8}, \tfrac{1}{4}),\ (\tfrac{3}{4}, \tfrac{3}{2}),\ (\tfrac{1}{8}, \tfrac{1}{4}) \right)$. For example, consider the allocation

$$\left((x^1, y^1), (x^2, y^2), (x^3, y^3), (x^4, y^4) \right) = \left((\tfrac{1}{6}, \tfrac{1}{6}),\ (\tfrac{5}{6}, \tfrac{5}{6}),\ (\tfrac{1}{6}, \tfrac{1}{6}), (\tfrac{5}{6}, \tfrac{5}{6}) \right).$$

The S-allocation $(\widetilde{x}^i, \widetilde{y}^i)_S$ makes each consumer strictly better off. So *any* allocation to N that gives each of the odd consumers $\frac{1}{6}$ unit of each good, or less, is not in the core.

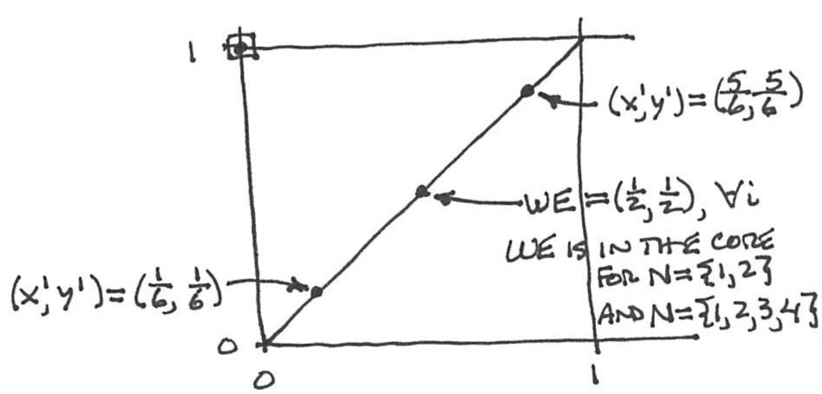

Figure 6

Exercise 1: Exactly which allocations of the form

$$((x^1, y^1), (x^2, y^2), (x^3, y^3), (x^4, y^4)) = ((a, a), (b, b), (a, a), (b, b))$$

are in the core of the four-consumer economy?

Exercise 2: Now assume there are r consumers of each type, with the utility function and initial bundles in the example. Determine which allocations of the form in Exercise 1 are core allocations. This requires two steps. The algebra in Step 1 is more difficult; but with Step 1 in hand, Step 2 is easy.

Step 1: Represent each coalition S by the pair (r, k), where r is the number of odd members and k is the number of even members. You need to show that the coalitions that can improve upon the most allocations are the ones in which $k = r - 1$.

Step 2: Determine for each r what is the smallest value of a for which the allocation in Exercise 1 can't be improved upon by a coalition with composition $(r, r - 1)$.

Notice how quickly the core becomes very small, if the only core allocations are the ones that have the form in Exercise 1. (And these are indeed the only core allocations, as we'll see shortly.)