Exercise: The Core Shrinks Under Replication

We begin with a 2×2 "Edgeworth Box" exchange economy: each consumer has the same preference, described by the utility function $u(x, y) = xy$; Consumer 1 owns the bundle $(\mathring{x}_1, \mathring{y}_1) = (15, 30)$; and Consumer 2 owns the bundle $(\mathring{x}_2, \mathring{y}_2) = (75, 30)$.

(a) Verify that there is a unique Walrasian (competitive) equilibrium, in which the price ratio is $p_x/p_y = 2/3$ and the consumption bundles are $(x_1, y_1) = (30, 20)$ and $(x_2, y_2) = (60, 40)$.

(b) Verify that the Pareto allocations are the ones that allocate the entire resource endowment of $(\mathring{x}, \mathring{y}) = (90, 60)$ and satisfy $y_1/x_1 = y_2/x_2 = 2/3$.

(c) In the Edgeworth Box draw the competitive allocation, the Pareto allocations, and each consumer's budget constraint at the competitive prices. Draw each consumer's indifference curve containing his initial bundle and indicate the core allocations in the diagram.

(d) Verify that the Pareto allocations for which $x_1 < \sqrt{675}$ are not in the core. Note that $\sqrt{675}$ is approximately 26. Similarly, the Pareto allocations for which $x_2 < \sqrt{3375} \approx 58.1$ are not in the core.

(e) Consider a proposed allocation $(\hat{x}_1, \hat{y}_1) = (27, 18)$ and $(\hat{x}_2, \hat{y}_2) = (63, 42)$. Note that each consumer's marginal rate of substitution at the proposal is 2/3. Verify that the proposal is in the core. Verify that the "trading ratio" τ defined by the proposal is $\tau = 1$. As in our lecture notes on the Debreu-Scarf Theorem, use the "shrinkage factor" $\lambda_1 = 2/3$ and the "expansion factor" $\lambda_2 = 4/3$ to verify that a coalition of just two "Type 1" consumers and one "Type 2" consumer can unilaterally allocate their initial bundles to make all three of them better off than in the proposal. Therefore the proposal is not in the core if there are two or more consumers of each type.

(f) Now consider the proposal $(\hat{x}_1, \hat{y}_1) = (28\frac{1}{2}, 19)$ and $(\hat{x}_2, \hat{y}_2) = (61\frac{1}{2}, 41)$, and use the same λ_1 and λ_2 as in (e) to establish that this proposal too is not in the core if there are two or more consumers of each type.

(g) Now consider the proposal $(\hat{x}_1, \hat{y}_1) = (29, 19\frac{1}{3})$ and $(\hat{x}_2, \hat{y}_2) = (61, 40\frac{2}{3})$, and use the factors $\lambda_1 = 4/5$ and $\lambda_2 = 6/5$ to establish that this proposal is not in the core if there are three or more consumers of each type.