Large Economies: The Replication Model and Equal Treatment Allocations

Example 3 suggests that the core may be much smaller in a very large economy than it is when the economy has only a few consumers: with many consumers, there are more possibilities for coalitions to improve on proposed allocations. So perhaps, if the economy is large, the core will consist of only the Walrasian allocation(s)— which we know are in the core — and allocations extremely close to the Walrasian allocation(s).

Modeling this idea in a satisfactory way is not easy — for example, the dimension of the space of allocations depends on the number of consumers, so it's not clear how to represent the size of the set of core allocations for economies with different numbers of consumers. We tackle this problem with a very special way of modeling economies with different numbers of consumers: larger economies are modeled as replications of a smaller, basic economy.

The Basic Economy:

We start with a **basic economy** $E = (u^t, \mathring{\mathbf{x}}^t)_{t=1}^T$, which consists of T consumers. We refer to each of these consumers as one of the **types** that will be replicated. (Thus, the index t stands for **type**; and T is the set of all types as well as the number of types.) An allocation in E is, as usual, a T-tuple of bundles, $(\mathbf{x}^t)_T \in \mathbb{R}_+^{T\ell}$ — a bundle for each consumer in T.

The r-fold Replication of E:

In the **r-fold replication of** E, which we'll denote by $r * E$, there are r copies of each of the consumer types in the basic economy E. Thus, the economy $r * E$ has rT consumers, and we have to index them by indicating which type *and* which copy we're referring to:

the rT consumers in $r * E$ are indexed by $(t, q) \in r * T := T \times \{1, \ldots, r\}$.

Equivalently, $r * T = \{(t, q) \mid t = 1, \ldots, T;\ q = 1, \ldots, r\}$. An allocation in $r * E$ is an rT-tuple of bundles, $(\mathbf{x}^{tq})_{r*T} \in \mathbb{R}_+^{rT\ell}$.

Allocations in $r * E$ that give the same bundle to every consumer of a given type will play a central role in the analysis. Because consumers of the same type get the same bundle in such an allocation, we refer to these allocations as *equal-treatment allocations*:

Definition: An **equal-treatment allocation** (abbreviated **ETA**) in $r * E$ is an allocation that satisfies the condition $\forall t, q, q' : \mathbf{x}^{tq} = \mathbf{x}^{tq'}$.

Example 3 was a replication economy in which $T = 2$ and $r = 2$. In $r * E$, all four consumers' utility functions were the same (*i.e.*, the two types, $t = 1$ and $t = 2$, happened to have the same utility function, although this would not be the case in general); and $\mathring{\mathbf{x}}^{11} = \mathring{\mathbf{x}}^{12} = (0, 1)$ and $\mathring{\mathbf{x}}^{21} = \mathring{\mathbf{x}}^{22} = (1, 0)$. In the example, we checked some ETAs (the ones that were replications of core allocations in the basic economy E) to determine which ones might be in the core of the replication economy $r * E$. We will show that *only* ETAs can be core allocations — *i.e.*, the core consists only of ETAs — so that the ETAs are the only allocations we need to check in order to determine whether they can be improved upon by some coalition. All other allocations *can* be improved upon.

First we have a useful remark and proposition:

Remark: If $(\mathbf{x}^{tq})_{r*T}$ is an ETA, then $\sum_{t=1}^{T} \sum_{q=1}^{r} \mathbf{x}^{tq} = \sum_{t=1}^{T} r\mathbf{x}^t$, where $\mathbf{x}^t = \mathbf{x}^{tq}, q = 1, \ldots r$. In particular, $(\mathring{\mathbf{x}}^{tq})_{r*T}$, the initial allocation in $r * E$, is an ETA, and therefore any feasible allocation (even if it's *not* an ETA) must satisfy

$$\sum_{t=1}^{T} \sum_{q=1}^{r} \mathbf{x}^{tq} \leqq \sum_{t=1}^{T} \sum_{q=1}^{r} \mathring{\mathbf{x}}^{tq} = \sum_{t=1}^{T} r\mathring{\mathbf{x}}^t = r \sum_{t=1}^{T} \mathring{\mathbf{x}}^t.$$

Given any allocation $(\mathbf{x}^{tq})_{r*T}$ in $r * E$, let $\overline{\mathbf{x}}^t$ denote the mean bundle the consumers of type t receive in $(\mathbf{x}^{tq})_{r*T}$:

$$\overline{\mathbf{x}}^t := \frac{1}{r} \sum_{q=1}^{r} \mathbf{x}^{tq}.$$

The following proposition tells us that if the allocation $(\mathbf{x}^{tq})_{r*T}$ is feasible, then any coalition S consisting of exactly one consumer of each type can unilaterally obtain the allocation $(\overline{\mathbf{x}}^t)_T$:

Proposition: If $(\mathbf{x}^{tq})_{r*T}$ is feasible for $r * T$ in $r * E$, then $(\overline{\mathbf{x}}^t)_T$ is feasible for T.

Proof:

$$\begin{aligned}
\sum_{t=1}^{T} \overline{\mathbf{x}}^t &= \sum_{t=1}^{T} \frac{1}{r} \sum_{q=1}^{r} \mathbf{x}^{tq} \\
&= \frac{1}{r} \sum_{t=1}^{T} \sum_{q=1}^{r} \mathbf{x}^{tq} \\
&\leqq \frac{1}{r}\, r \sum_{t=1}^{T} \mathring{\mathbf{x}}^{tq}, \text{ by the Remark above, because } (\mathbf{x}^{tq})_{r*T} \text{ is feasible} \\
&= \sum_{t=1}^{T} \mathring{\mathbf{x}}^t. \qquad \|
\end{aligned}$$

Theorem: Let $E = (u^t, \mathring{\mathbf{x}}^t)_{t=1}^T$ be an economy in which each u^t is strictly quasiconcave, and let $r \in \mathbb{N}$. Then every core allocation in $r * E$ is an equal-treatment allocation.

Proof:

Let $(\mathbf{x}^{tq})_{r*T}$ be a feasible allocation for $r * E$. Wlog, suppose that for each type t, the first copy ($q = 1$) is treated the worst in $(\mathbf{x}^{tq})_{r*T}$ — *i.e.*,

$$\forall (t, q) \in r * T: \; u^t(\mathbf{x}^{t1}) \leqq u^t(\mathbf{x}^{tq}).$$

(Note that we haven't yet assumed that $(\mathbf{x}^{tq})_{r*T}$ is *not* an ETA: the inequality above could be satisfied weakly for all q.)

Let S be the coalition consisting of all the first-copy consumers: $S = \{(1,1), (2,1), \ldots, (T,1)\}$. We have shown in the proposition above that the T-tuple of mean bundles $\overline{\mathbf{x}}^t$ is a feasible allocation for S. Moreover, for each type t, the mean bundle $\overline{\mathbf{x}}^t$ is a convex combination of the r bundles $\mathbf{x}^{t1}, \mathbf{x}^{t2}, \ldots, \mathbf{x}^{tr}$, all of which lie in the upper-contour set of $\mathbf{x}^{t1}$. Since the upper-contour set is convex, $\overline{\mathbf{x}}^t$ also lies in the upper-contour set — *i.e.*, $u^t(\overline{\mathbf{x}}^t) \geqq u^t(\mathbf{x}^{tq})$ for $q = 1, \ldots, r$.

Now suppose that $(\mathbf{x}^{tq})_{r*T}$ is not an ETA, and let t be any type for which all copies do not receive identical bundles — *i.e.*, there are some copies q and q' for which $\mathbf{x}^{tq} \neq \mathbf{x}^{tq'}$. Since all r coefficients are strictly positive when expressing $\overline{\mathbf{x}}^t$ as a convex combination of the bundles $\mathbf{x}^{tq}$ for $q = 1, \ldots, r$, and since u^t is strictly quasiconcave and all the bundles $\mathbf{x}^{tq}$ lie in the upper-contour set of $\mathbf{x}^{t1}$, we have $u^t(\overline{\mathbf{x}}^t) > u^t(\mathbf{x}^{t1})$. This establishes that the coalition S can unilaterally improve upon the non-ETA allocation $(\mathbf{x}^{tq})_{r*T}$. $\|$

Figures 1 and 2 depict the argument in the proof. In the figures, there are $r = 3$ copies of each type of consumer, and in each figure the copy $q = 1$ has the smallest utility level. In Figure 1 we have $u^1(\mathbf{x}^{11}) = u^1(\mathbf{x}^{12}) < u^1(\mathbf{x}^{13})$. In Figure 2 all three copies of type $t = 2$ have the same utility: copy $q = 1$ is no worse off than any other copy — his utility level is smallest, but it's not *smaller*.

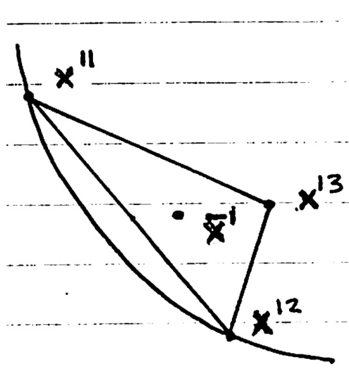

Figure 1

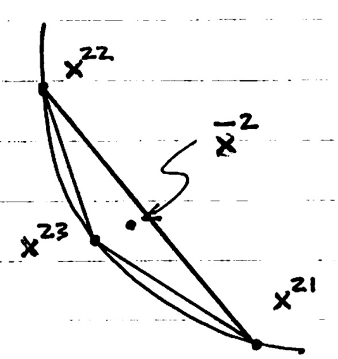

Figure 2

Figures 1 and 2 are intended only to show how the assumption of strictly convex preferences enters into the proof of the theorem. But it's patently clear on other grounds that the bundles in Figures 1 and 2 can't be part of a core allocation: the bundles assigned to various consumers of a given type have different MRS's and therefore can't be part of a Pareto allocation, and *a fortiori* they can't be part of a core allocation.

Here's an example where it's not at all obvious that the allocation can't be in the core. In order to show it, you need to essentially make the argument, for the example, that's in the proof of the theorem — or, of course, once we have the theorem, simply apply it to the allocation: since the allocation's not an ETA, it can't be in the core.

Example: There are two goods, two types, and two consumers of each type. The Type 1 consumers are both endowed with the bundle $(2,4)$ and the Type 2 consumers are both endowed with the bundle $(6,1)$; the economy's total endowment is therefore the bundle $(16,10)$. The proposed allocation is

$$\text{Type 1's:} \quad \mathbf{x}^{11} = (3,3), \quad \mathbf{x}^{12} = (6,1); \qquad \text{Type 2's:} \quad \mathbf{x}^{21} = (2,4), \quad \mathbf{x}^{22} = (5,2).$$

It's useful to see the same information presented in the following table:

	$\mathring{\mathbf{x}}^t$	$\mathbf{x}^{t1}$	$\mathbf{x}^{t2}$	$\overline{\mathbf{x}}^t$
$t=1$	$(2,4)$	$(3,3)$	$(6,1)$	$(4.5,2)$
$t=2$	$(6,1)$	$(2,4)$	$(5,2)$	$(3.5,3)$
$\Sigma_{t=1}^{2}$	$(8,5)$	$(5,7)$	$(11,3)$	$(8,5)$

Utility functions (or preferences) haven't been specified for the two types. Therefore it isn't clear, for either type, which of the two consumers of the type is worse off. You don't need to know which one is worse off to *apply* the theorem, but you would obviously have to know the preferences (or reproduce the argument in the proof) in order to verify directly that the proposed allocation isn't in the core.

The following exercise demonstrates why it's not obvious, without our theorem, that the proposed allocation isn't in the core (unlike in Figures 1 and 2, where the MRS's aren't all equal).

Exercise: Provide a geometric argument to show, in the above example, that there exist representable preferences (and therefore utility functions) in which the first consumer of each type is strictly worse off than the second consumer at the proposed allocation and all consumers have the same MRS (therefore the proposal is Pareto optimal — the four-player "coalition of the whole" can't improve on it). The same argument also shows that there exist preferences in which all consumers have the same MRS and the second consumer of each type is strictly worse off than the first consumer. **Hint:** In the first case it's clear that the common MRS must satisfy $MRS > 2/3$ and in the second case it's clear that we must have $MRS < 2/3$. It's also clear that in the $MRS > 2/3$ case the proposal can also be individually rational, so that none of the one-trader coalitions can improve on it either (by simply consuming his or her endowment bundle).