The Core:
Edgeworth's "Recontracting" or Bargaining Equilibrium

If there are only two consumers in the market, each one's potential to influence the prices is obvious. It's therefore unsatisfactory to assume, as the Walrasian equilibrium does, that each consumer ignores this potential and simply behaves as a price-taker. Edgeworth asked what the outcome would be if we *don't* assume that markets and prices are used. Suppose we assume only that resources are privately owned and controlled and that the traders will "bargain" with one another. Can we say anything about which allocations will or will not occur? In other words, which allocations could be viable *equilbria* of a bargaining process? Edgeworth analyzed this question for two goods and an arbitrary number of traders. Debreu & Scarf generalized to any number of goods, and others have generalized the Debreu & Scarf result.

Just as we abstracted away from the dynamics of changing prices when we analyzed market equilibrium, here we abstract away from the dynamic process of bargaining, focusing only on those situations from which no (further) bargaining will occur. And of course we begin with the simplest case, the "Edgeworth Box" situation: exchange only (no production), two goods, and two people.

At what allocations will the traders agree to trade, thereby moving to a different allocation? A first answer: they will trade if they can both gain from trading — *i.e.*, if a Pareto improvement exists. Therefore, an "equilibrium" must at least be Pareto efficient. Are *all* Pareto efficient allocations equilibria, so that the two notions of bargaining equilibrium and Pareto efficiency coincide? No: neither trader will agree to a proposed allocation that makes him worse off than he would have been if he had not traded — *i.e.*, to an allocation that makes him worse off than he is at the bundle he owns initially.

In the two-trader case, then, a reasonable definition of bargaining equilibrium is as follows:

An allocation $\hat{\mathbf{x}} = (\hat{\mathbf{x}}^i)_{i \in N}$ is a bargaining equilibrium for $N = \{1, 2\}$ if

(1) $\hat{\mathbf{x}}$ is Pareto efficient, and

(2) For each i: $u^i(\hat{\mathbf{x}}^i) \geqq u^i(\mathring{\mathbf{x}}^i)$.

An allocation that satisfies (2) is said to be *individually acceptable* or *individually rational*.

The bargaining equilibria in the two-trader case are exactly the allocations that Edgeworth called the "recontracting" equilibria. Today we call them the *core* allocations. In the Edgeworth Box they form the "contract curve." (Many economists use the term "contract curve" to mean the locus of *all* the Pareto allocations in the box, typically a much larger set.) Note that in this two-person case, with increasing preferences, every Walrasian equilibrium allocation is in the core (because a Walrasian allocation is Pareto efficient and individually rational), but almost all core allocations are *not* Walrasian equilibria for the given distribution of initial ownership.

Example 1: On the following page is a simple example in which the core of a two-person economy is determined.

In the definition of the two-person core given above, we can paraphrase Conditions (1) and (2) as follows:

(1) There is no way for the two individuals together to improve upon the allocation $\hat{\mathbf{x}}$, and
(2) There is no way for either individual, by himself, to unilaterally improve upon $\hat{\mathbf{x}}$.

Now suppose there are n consumers, indexed by $i \in N = \{1, \ldots, n\}$, with utility functions u^i and initial bundles $\mathring{\mathbf{x}}^i$. We say that the *core* is the set of all allocations that are feasible and cannot be improved upon by *any* "coalition" of members of N.

Definition 1: A *coalition* is a nonempty subset of N. Let $\mathbf{x} = (\mathbf{x}^i)_{i \in N} \in \mathbb{R}^{nl}_+$ be an allocation. A coalition S can *unilaterally improve upon* $\mathbf{x}$ if there is an allocation to S — say, $(\tilde{\mathbf{x}}^i)_{i \in S}$ — that is both
(a) feasible for S: $\sum_{i \in S} \tilde{\mathbf{x}}^i \leqq \sum_{i \in S} \mathring{\mathbf{x}}^i$ and
(b) a Pareto improvement for S: $u_i(\tilde{\mathbf{x}}^i) \geqq u^i(\mathbf{x}^i)$ for all $i \in S$, and $u_i(\tilde{\mathbf{x}}^i) > u^i(\mathbf{x}^i)$ for some i.

Definition 2: The *core* of an economy is the set of all allocations that are feasible and that cannot be improved upon by any coalition.

In the language of Definition 1, the Pareto allocations are the feasible allocations that can't be improved upon by the coalition of *all* individuals, and the individually rational allocations are the ones that can't be unilaterally improved upon by any of the *one-person* coalitions. Therefore all core allocations are both Pareto efficient and individually rational, just as in the two-person case. But as Example 2 below shows, when $n > 2$ there are generally allocations that are both Pareto efficient and individually rational but which are nevertheless *not* in the core — because they can be improved upon by some intermediate-sized coalition, one that includes more than one person but not everyone.

A Simple Core Example

$u_A(x_A, y_A) = x_A y_A$ $\qquad$ $u_B(x_B, y_B) = x_B y_B$

$(\mathring{x}_A, \mathring{y}_A) = (10, 90)$ $\qquad$ $(\mathring{x}_B, \mathring{y}_B) = (90, 10)$

$\mathring{u}_A = 900$ $\qquad$ $\mathring{u}_B = 900$

The Core:

$MRS_A = MRS_B$ $\qquad$ $u_A \geqq 900$ $\qquad$ $u_B \geqq 900$

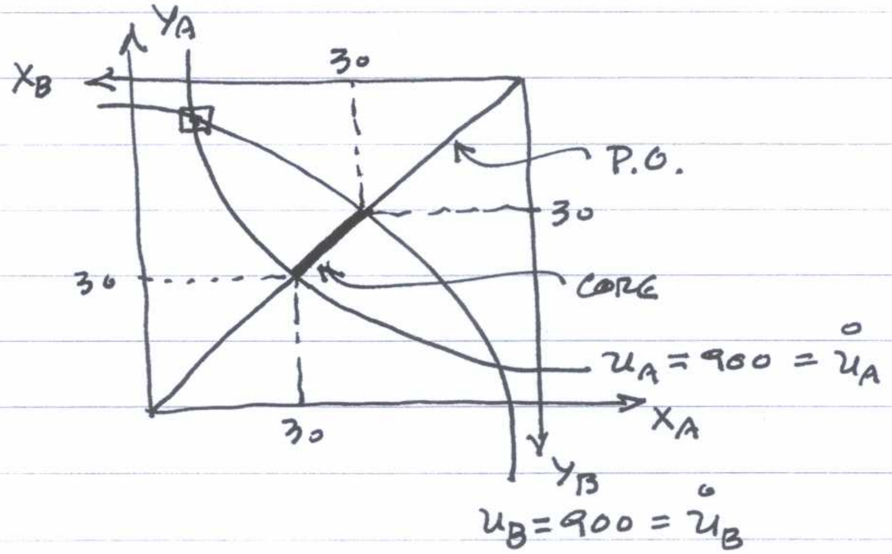
$\Downarrow$

$x_A = y_A, \ \therefore u_A = x_A^2$ $\qquad$ $\therefore x_A^2 \geqq 900, \ x_A \geqq 30$

$x_B = y_B, \ \therefore u_B = x_B^2$ $\qquad$ $\therefore x_B^2 \geqq 900, \ x_B \geqq 30$

Summarizing: $x_A = y_A$; $x_A + x_B = 100$, $y_A + y_B = 100$;

$30 \leqq x_A \leqq 70$.



Example 2: This example shows that when $n > 2$ the requirement that *no* coalition be able to improve upon $(\hat{\mathbf{x}}^i)_1^n$ is generally a stronger requirement than merely requiring Pareto efficiency ("N cannot improve") and individual rationality ("no {i} can improve").

$$N = \{1, 2, 3\} \qquad \forall i \in N : u^i(\mathbf{x}^i) = x_1^i x_2^i$$

$$\begin{array}{rcll}
\mathring{\mathbf{x}}^1 & = & (19, 1) & \therefore \mathring{u}^1 = 19 \\
\mathring{\mathbf{x}}^2 & = & (1, 19) & \therefore \mathring{u}^2 = 19 \\
\mathring{\mathbf{x}}^3 & = & (10, 10) & \therefore \mathring{u}^3 = 100 \\
\hline
\sum \mathring{\mathbf{x}}^i & = & (30, 30) &
\end{array}$$

The following allocation $(\hat{\mathbf{x}}^i)_1^3$ is Pareto efficient and individually rational:

$$\begin{array}{rcccl}
\hat{\mathbf{x}}^1 & = & \hat{\mathbf{x}}^2 = & (9, 9) & \therefore \hat{u}^1 = \hat{u}^2 = 81 \\
\hat{\mathbf{x}}^3 & = & & (12, 12) & \therefore \hat{u}^3 = 144 \\
\hline
\sum \hat{\mathbf{x}}^i & = & & (30, 30) &
\end{array}$$

But the coalition $S = \{1, 2\}$ can unilaterally improve upon $(\hat{\mathbf{x}}^i)_1^3$ as follows:

$$\tilde{\mathbf{x}}^1 = \tilde{\mathbf{x}}^2 = (10, 10) \qquad \therefore \tilde{u}^1 = \tilde{u}^2 = 100 > 81.$$

Therefore $(\hat{\mathbf{x}}^i)_1^3$ is not in the core.

Exercise: Exercise #5.1 in the Exercise Book asks you to find all the core allocations for this example.