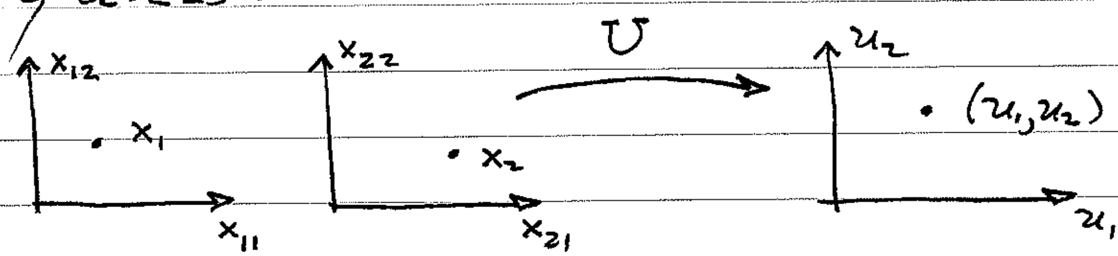
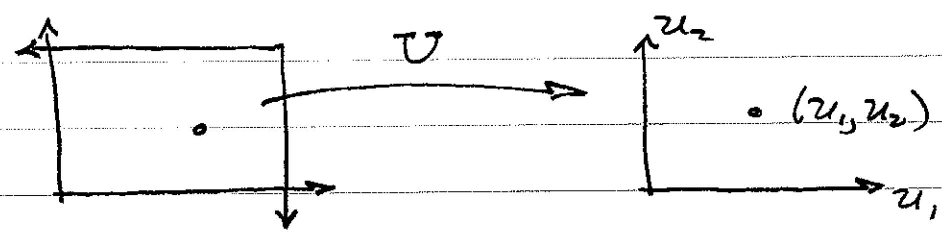
# THE UTILITY FRONTIER

ANY ALLOCATION $(x_i)_N$ TO A SET $N = \{1, \dots, n\}$ OF INDIVIDUALS WITH UTILITY FUNCTIONS $u_i(\cdot)$, $i \in N$, YIELDS A PROFILE $(u_1, \dots, u_n)$ OF RESULTING UTILITY LEVELS:

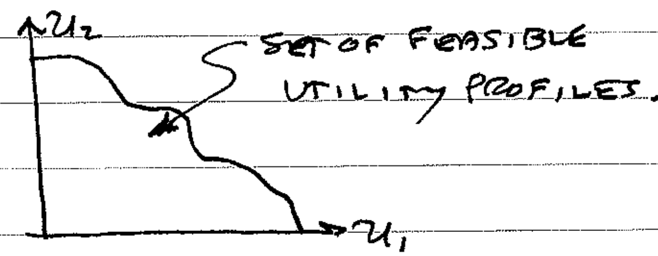



(OR)



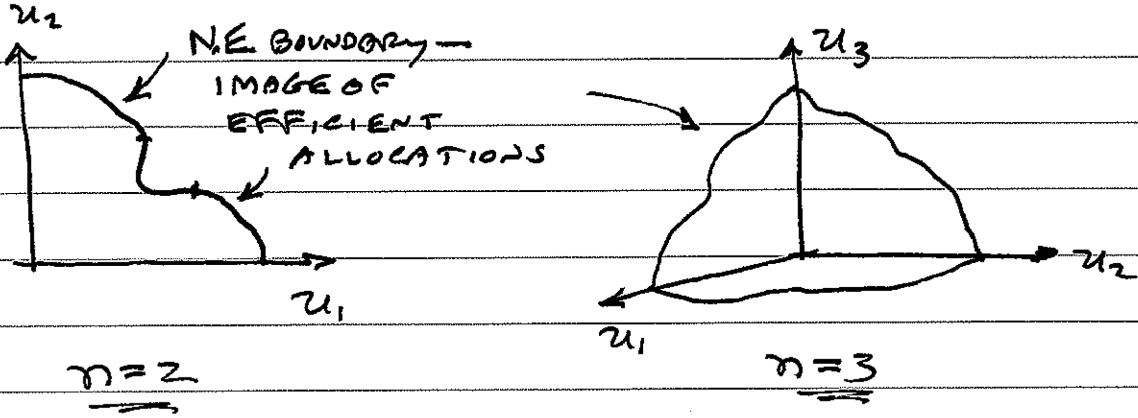

$$U((x_i)_N) = (u_1(x_1), \dots, u_n(x_n)) \in \mathbb{R}^n.$$

THE IMAGE UNDER $U(\cdot)$ OF THE SET OF ALL FEASIBLE ALLOCATIONS (THOSE THAT SATISFY $\sum_{i \in N} x_i \leqq \mathring{x}$) IS THE SET OF FEASIBLE UTILITY PROFILES:




AN ALLOCATION IS CLEARLY PARETO EFFICIENT IF AND ONLY IF IT IS MAPPED BY $U(\cdot)$ TO THE "NORTHEAST" PART OF THE BOUNDARY OF THE SET OF FEASIBLE UTILITY PROFILES, THE <u>UTILITY FRONTIER</u>.

## THE UTILITY FRONTIER




HOW CAN WE CALCULATE THE UTILITY FRONTIER? IT WILL BE IN THE FORM OF A FUNCTION — e.g., $u_1 = f(u_2, \ldots, u_n)$ — THAT TELLS, FOR GIVEN UTILITY LEVELS OF $n-1$ INDIVIDUALS, WHAT IS THE HIGHEST UTILITY LEVEL THAT'S FEASIBLE FOR THE REMAINING INDIVIDUAL.

IN FACT, THE UTILITY FRONTIER IS SIMPLY THE GRAPH OF THE <u>VALUE FUNCTION</u> ASSOCIATED WITH THE PARETO EFFICIENCY MAXIMIZATION PROBLEM — i.e., THE FUNCTION THAT GIVES THE OBJECTIVE VALUE $u_1$ AS A FUNCTION OF THE RHS-VALUES $\bar{u}_2, \ldots, \bar{u}_n$ IN THE PROBLEM

$$\max_{(x_i)_N} u_1(x_1) \text{ s.t. } x_{ik} \geqq 0 \quad (\forall i, k)$$

$$\text{AND s.t. } \sum_N x_i \leqq \mathring{x}$$

$$\text{AND s.t. } u_2(x_2) \geqq \bar{u}_2$$

$$\vdots \qquad \vdots$$

$$u_n(x_n) \geqq \bar{u}_n .$$

# Solution Function & Value Function

$$\max_x f(x;\alpha)$$

Solution function: $x = x(\alpha)$.

Value function: $v(\alpha) := f(x(\alpha))$.

**Examples:**

① Consumer max problem; Demand Theory:

$$\max_x u(x) \text{ s.t. } p\cdot x \leqq w \qquad \alpha = (p;w)$$

Sol'n fn. is demand fn.: $x(p;w)$

Value fn. is indirect utility fn: $v(p;w)$

② Firm's exp. cost-minimization problem:

$$\min_x E(x) = w\cdot x \text{ s.t. } f(x) \geqq y \qquad \alpha = (w;y)$$

Sol'n fn. is input demand fn: $x(w;y)$

Value fn. is cost fn: $c(y;w)$.

③ Utility frontier & Pareto problem:

$$\max_{x\in\mathbb{R}_+^{nl}} u_1(x) \text{ s.t. } u_2(x),\ldots,u_n(x) \geqq \bar{u}_2,\ldots,\bar{u}_n.$$

!

SOLUTION FUNCTION: $x(\bar{u}_2, \dots, \bar{u}_n)$, PARETO SET OF ALLOCATIONS

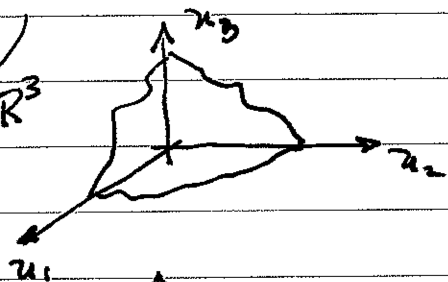
VALUE FUNCTION: $u_1(\bar{u}_2, \dots, \bar{u}_n)$, UTILITY FRONTIER

... IF IN IMPLICIT FUNCTION FORM:

$$g(u_1, u_2, \dots, u_n) = \xi .$$

... A SURFACE IN $\mathbb{R}^3$

2-DIM'L




DOESN'T HAVE TO BE IN POSITIVE ORTHANT, BUT WILL GEN'LY HAVE LOWER BOUND IN EACH DIMENSION.

$\xi$

EXAMPLE:

$N = \{1, \ldots, n\}$; $u_i(x_i, y_i) = x_i y_i$, $\forall i \in N$;

TOTAL ENDOWMENT IS $(\mathring{x}, \mathring{y})$.

PARETO EFFICIENCY REQUIRES THAT, FOR SOME NUMBER $r$:

$$\frac{y_1}{x_1} = \frac{y_2}{x_2} = \cdots = \frac{y_n}{x_n} = r; \quad \text{i.e., } y_i = r x_i, \ \forall i \in N.$$

$$\therefore \ \mathring{y} = r \mathring{x} \quad \left[\mathring{y} = \Sigma y_i = \Sigma r x_i = r \Sigma x_i = r \mathring{x}\right],$$

$$\text{i.e., } \boxed{r = \frac{\mathring{y}}{\mathring{x}}}$$

AT ANY EFFICIENT ALLOCATION, THEN, WE MUST HAVE, $\forall i \in N$:

$$u_i(x_i, y_i) = x_i y_i = (x_i)(r x_i) = r x_i^2;$$

$$\text{i.e., } \sqrt{u_i} = \sqrt{r}\, x_i.$$

$$\therefore \ \sum_{i \in N} \sqrt{u_i} = \sqrt{r} \sum_{i \in N} x_i = \sqrt{r}\, \mathring{x}.$$

$$\therefore \ \sum_{i \in N} \sqrt{u_i} = \sqrt{r}\, \mathring{x} = \frac{\sqrt{\mathring{y}}}{\sqrt{\mathring{x}}}\, \mathring{x} = \sqrt{\mathring{x}\mathring{y}}.$$

IN OTHER WORDS, THE UTILITY FRONTIER IS THE EQUATION

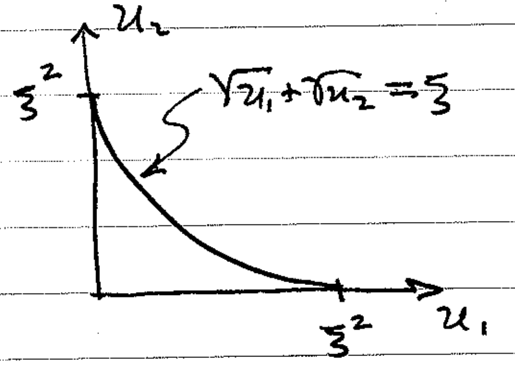
$$\sum_{i \in N} \sqrt{u_i} = \sqrt{\mathring{x}\mathring{y}},$$

OR ITS GRAPH IN $\mathbb{R}^n$.

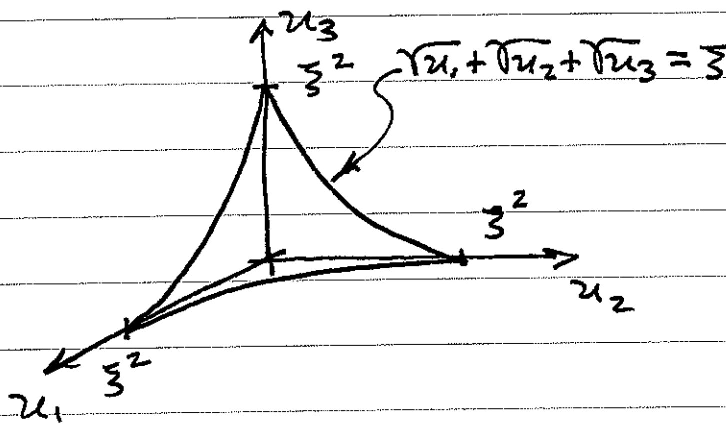
DANGER: THE UTILITY FRONTIER HAS THIS FORM IN THIS EXAMPLE, WHERE ALL UTILITY FUNCTIONS ARE OF THE FORM $u(x,y) = xy$.

Let $\zeta := \sqrt{\mathring{x}\mathring{y}}$.

<u>$n=2$:</u>




<u>$n=3$:</u>



EXAMPLE: (THE UTILITY FRONTIER AND THE CORE)

$N = \{1,2,3\}$; $u_i(x_{i1}, x_{i2}) = x_{i1} x_{i2}$, $i = 1,2,3$.

$\overset{\circ}{x}_1 = \overset{\circ}{x}_2 = (30,0)$; $\overset{\circ}{x}_3 = (0,60)$.

PROPOSAL: $\hat{x}_i = (20,20)$, $i=1,2,3$. $u_i(\hat{x}_i) = 400$, $i = 1,2,3$.

CLEARLY, $(\hat{x}_i)_N$ IS PARETO EFFICIENT AND INDIVIDUALLY ACCEPTABLE.

BUT $\{1,3\}$ CAN IMPROVE UPON $(\hat{x}_i)_N$ VIA $(\tilde{x}_i)_{\{1,3\}}$,

WHERE $\tilde{x}_1 = \tilde{x}_3 = (15,30)$:

WE HAVE $\tilde{x}_1 + \tilde{x}_3 = (30,60) = \overset{\circ}{x}_1 + \overset{\circ}{x}_3$ AND $u_1(\tilde{x}_1) = u_2(\tilde{x}_2) = 450$.

THE COALITION $\{2,3\}$ COULD IMPROVE IN THE SAME WAY.

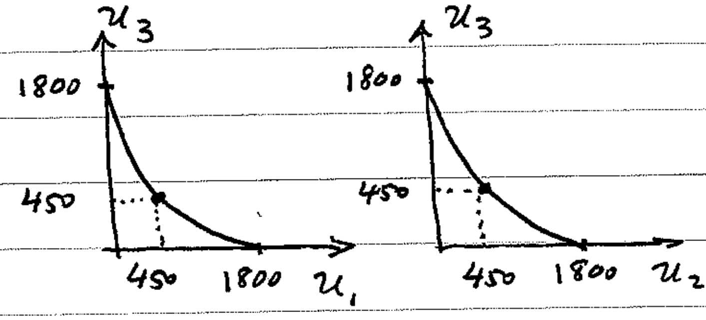
IN FACT, IT IS CLEAR THAT UNLESS A PROPOSAL $(x_i)_N$ GIVES BOTH $u_1 \geqq 450$ AND $u_2 \geqq 450$, OR ELSE $u_3 \geqq 450$, THEN EITHER $\{1,3\}$ OR $\{2,3\}$ WILL BE ABLE TO UNILATERALLY IMPROVE UPON $(x_i)_N$: ANY PROPOSAL THAT $u_3 < 450$ AND EITHER $u_1 < 450$ OR $u_2 < 450$ CAN BE IMPROVED UPON BY $\{1,3\}$ OR $\{2,3\}$ AS ABOVE.

IN FACT, THE UTILITY FRONTIERS FOR $\{1,3\}$ AND $\{2,3\}$ ARE $\sqrt{u_1} + \sqrt{u_3} = \sqrt{(30)(60)} = \sqrt{1800} = 30\sqrt{2}$

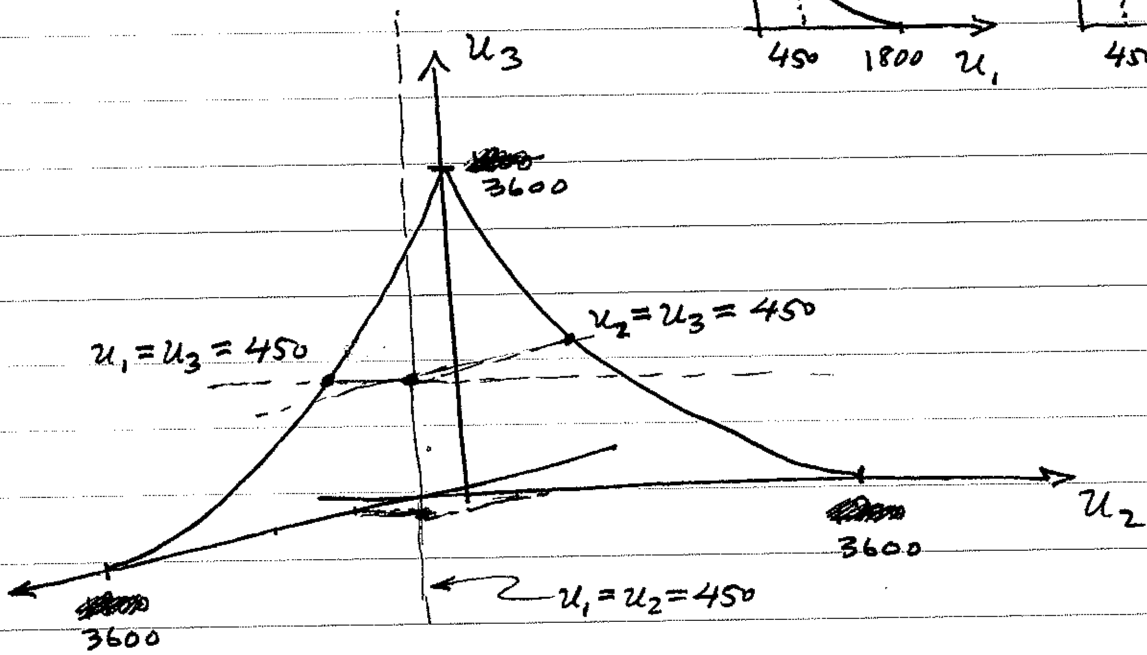
AND $\sqrt{u_2} + \sqrt{u_3} = \sqrt{(30)(60)} = \sqrt{1800} = 30\sqrt{2}$.

SINCE PARETO EFFICIENCY IN THIS EXAMPLE REQUIRES $x_{i1} = x_{i2} = z_i$, SAY, FOR $i = 1,2,3$, WE HAVE

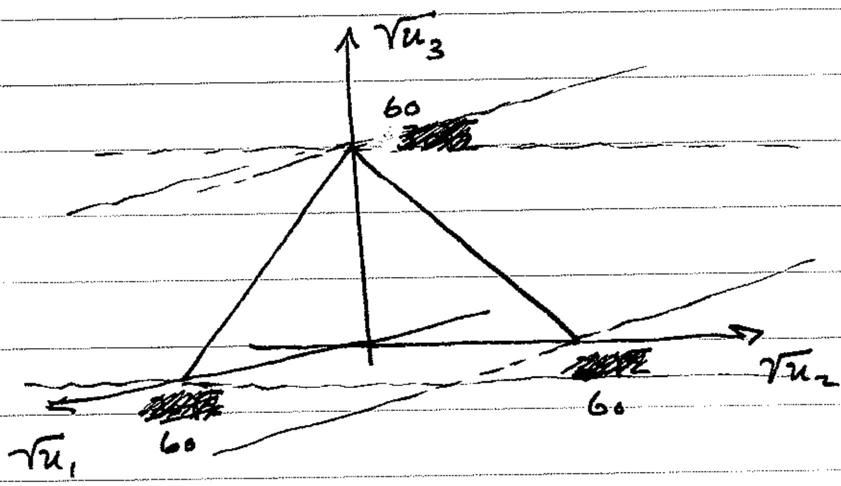
$$z_1 + z_3 \geqq 30\sqrt{2} \approx 42.4 \text{ AND } z_2 + z_3 \geqq 30\sqrt{2} \approx 42.4.$$



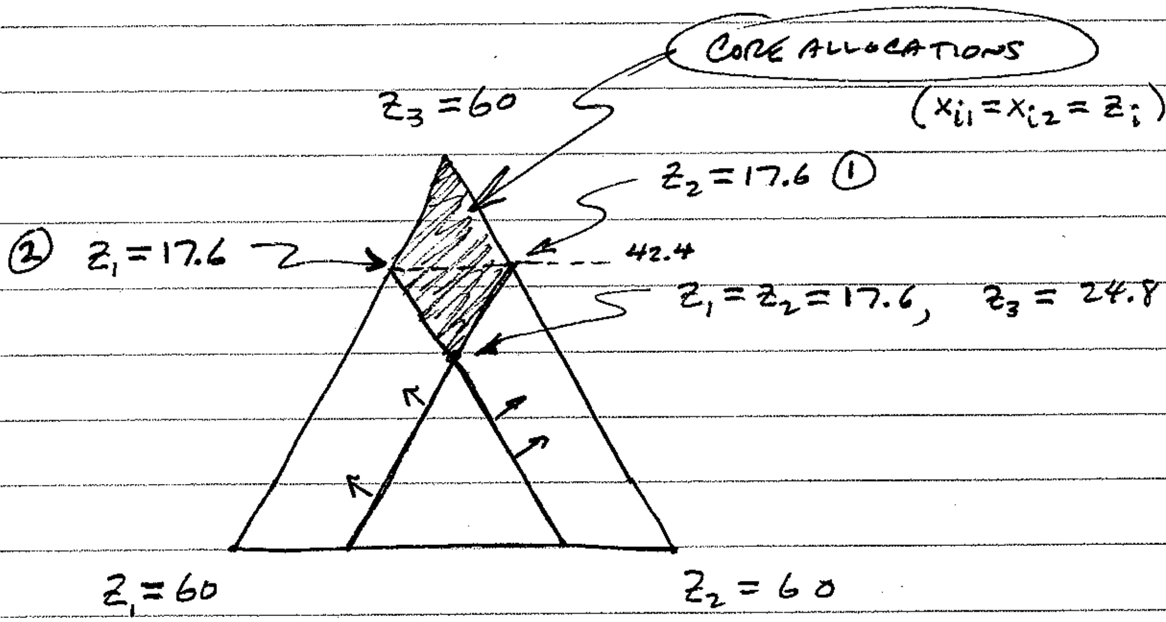
$u_3$
1800
450
450
1800
$u_1$
$u_3$
1800
450
450
1800
$u_2$

$u_3$
3600
$u_2 = u_3 = 450$
$u_1 = u_3 = 450$
$u_2$
3600
$u_1$
3600
$u_1 = u_2 = 450$

$\sqrt{u_3}$
60
$\sqrt{u_2}$
60
$\sqrt{u_1}$
60




① $z_1 + z_3 \geqq 42.4$ i.e., $z_2 \leqq 17.6$

② $z_2 + z_3 \geqq 42.4$ i.e., $z_1 \leqq 17.6$

$\therefore$ $z = (20, 20, 20)$ IS <u>NOT</u> IN THE CORE