Walrasian Equilibria are in the Core

In our examples, we've noted in every case that the Walrasian equilibrium allocation has been in the core — *i.e.*, it's been one of the possible "bargaining equilibria" of the economy. The theorem on the following page establishes that this is true under very general conditions: the consumers' preferences simply need to be locally nonsatiated. But this shouldn't be surprising: local nonsatiation is also the only assumption needed to establish the First Welfare Theorem, which says that if you can make a Pareto improvement on a proposed allocation, then the proposed allocation can't be the outcome of a Walrasian equilibrium. In fact, that's exactly the way our proof proceeded: we assumed that the proposed equilibrium allocation is not Pareto (*i.e.*, it can be improved upon), and then showed that the assumed improvement could not be feasible, a contradiction — the assumed improvement can't actually be accomplished.

To establish that a Walrasian allocation is more than just Pareto optimal, and in fact is actually in the core, we proceed in exactly the same way: we assume that the proposed allocation can be improved upon by *some* coalition — but not necessarily by the coalition consisting of all the traders — and show in the same way as before that the improvement could not be feasible for the coalition, using just its own resources. In other words, we show that an assumed improvement by *any* coalition can't actually be accomplished with the resources available to it, and therefore the proposal *is* in the core.

Note that the proof is identical to our proof of the First Welfare Theorem except that an arbitrary coalition S replaces the specific coalition N consisting of all traders, just as the above discussion suggests.

Theorem: Let $(\hat{p}, (\hat{x}^i)_N)$ be a competitive equilibrium for an economy $E = ((u^i, \mathring{x}^i))_N$, and assume that $\hat{p} \in \mathbb{R}^\ell_+$. If each u^i is locally nonsatiated (LNS), then $(\hat{x}^i)_N$ is in the core of E.

Proof:

Suppose that $(\hat{x}^i)_N$ is not in the core — i.e., there is a coalition S that can unilaterally improve upon $(\hat{x}^i)_N$ via an allocation $(\tilde{x}^i)_S$:

(a) $\sum_{i \in S} \tilde{x}^i \leqq \sum_{i \in S} \mathring{x}^i$

(b1) $\forall i \in S: u^i(\tilde{x}^i) \geqq u^i(\hat{x}^i)$,

(b2) $\exists i \in S: u^i(\tilde{x}^i) > u^i(\hat{x}^i)$.

Because $(\hat{p}, (\hat{x}^i)_N)$ is a competitive equilibrium, each $\hat{x}^i$ maximizes u^i on the budget set $\{x^i \in \mathbb{R}^\ell_+ \mid \hat{p} \cdot x^i \leqq \hat{p} \cdot \mathring{x}^i\}$. Therefore, (b2) ensures that

(c2) $\exists i \in S: \hat{p} \cdot \tilde{x}^i > \hat{p} \cdot \mathring{x}^i$,

and (b1) ensures that

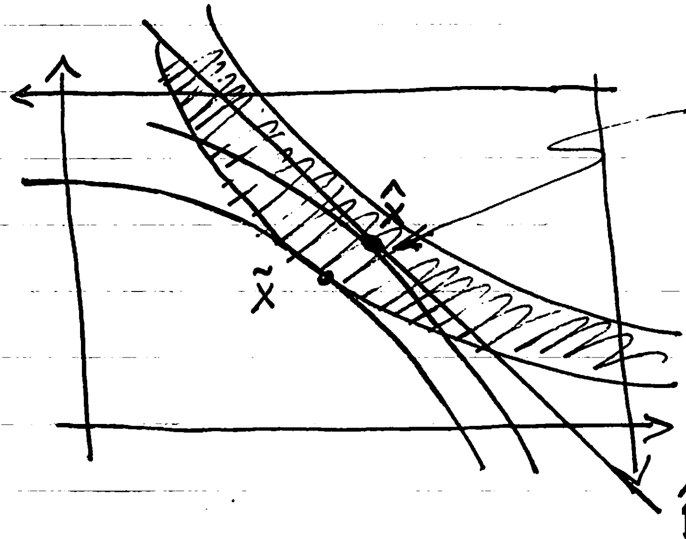
(c1) $\forall i \in S: \hat{p} \cdot \tilde{x}^i \geqq \hat{p} \cdot \mathring{x}^i$,

Duality [utility max. $\Rightarrow$ expenditure min.] yields $\hat{p} \cdot \tilde{x}^i \geqq \hat{p} \cdot \hat{x}^i$, but that's not enough if we don't also know that $\hat{p} \cdot \hat{x}^i \geqq \hat{p} \cdot \mathring{x}^i$.

as follows: If $\hat{p} \cdot \tilde{x}^i < \hat{p} \cdot \mathring{x}^i$, there would be a neighborhood η of $\tilde{x}^i$ for which $x^i \in \eta \Rightarrow \hat{p} \cdot x^i < \hat{p} \cdot \mathring{x}^i$, and by LNS such a nbd. contains an x^i that satisfies $u^i(x^i) > u^i(\tilde{x}^i) \geqq u^i(\hat{x}^i)$, which is inconsistent with $\hat{x}^i$ maximizing u^i on the budget set. Summing the inequalities in (c1) and (c2) over S, we have $\sum_S \hat{p} \cdot \tilde{x}^i > \sum_S \hat{p} \cdot \mathring{x}^i$ — i.e., $\hat{p} \cdot \sum_S \tilde{x}^i > \hat{p} \cdot \sum_S \mathring{x}^i$. Since $\hat{p} \in \mathbb{R}^\ell_+$, this yields $\sum_S \tilde{x}^i_k > \sum_S \mathring{x}^i_k$ for some k, which contradicts (a). ||

COUNTEREXAMPLE:

(TO SHOW THAT LNS IS ESSENTIAL)



THIS A WALRASIAN EQUIL'M, BUT $(\tilde{x}^i)_N$ IS A PARETO IMPROVEMENT; $\therefore (\hat{x}^i)_N$ IS NOT IN THE CORE.

THIS IS THE SAME EXAMPLE (MR. 1 HAS A THICK I-CURVE) AS FOR THE FIRST WELFARE THEOREM. NOTICE THAT THE PROOF GIVEN ABOVE IS ALSO VIRTUALLY THE SAME AS THE PROOF OF THE FIRST WELFARE THEOREM.