ARROW'S GE-WITH-SECURITIES MODEL

STATES: $s \in S = \{1, \ldots, S\}$ ONE GOOD (e.g., "DOLLARS")
CONSUMERS: $i \in N = \{1, \ldots, N\}$

CONSUMPTION PLANS: $(x_0^i, (x_s^i)) \in \mathbb{R}_+ \times \mathbb{R}_+^S = \mathbb{R}_+^{1+S}$

CONSUMERS: $(u^i; (\mathring{x}_0^i, (\mathring{x}_s^i)))$, $u^i : \mathbb{R}_+^{1+S} \to \mathbb{R}$,

SECURITIES:
INDEXED BY $k \in \{1, \ldots, K\} = K$
FOR EACH $k \in K$:

$$d_k = (d_{sk})_{s \in S} = \begin{bmatrix} d_{1k} \\ \vdots \\ d_{Sk} \end{bmatrix},$$ DIVIDENDS/RETURNS VECTOR.

y_k^i IS AMOUNT OF SECURITY k BOUGHT/HELD BY i.
i's PORTFOLIO: $(y_1^i, \ldots, y_K^i)$
q_k IS THE PRICE OF SECURITY k.

$D = [d_1, \ldots, d_K] = [d_{sk}]_{s \in S, k \in K}$, THE $S \times K$ MATRIX OF SECURITIES RETURNS.

NOTE THAT ANY d_{sk} OR y_k^i CAN BE POSITIVE, NEGATIVE, OR ZERO.

THE LINK BETWEEN SECURITIES AND PLANS:

$$x_s^i = \overset{\circ}{x}{}_s^i + \sum_{k\in K} y_k^i d_{sk}, \ \forall s \in S$$

i.e., $(x_s^i)_S = (\overset{\circ}{x}{}_s^i)_S + Dy^i \in \mathbb{R}_+^S$

LET $z^i = x^i - \overset{\circ}{x}{}^i$

i.e., $(x_s^i - \overset{\circ}{x}{}_s^i)_S = Dy^i \in \mathbb{R}_+^S$ $z = Dy$

EXAMPLES:

① CREDIT ONLY: $(K=1)$

$$x_H^i = \overset{\circ}{x}{}_H^i + (1+r)y^i \quad z = y\begin{bmatrix}1+r\\1+r\end{bmatrix}$$

$$x_L^i = \overset{\circ}{x}{}_L^i + (1+r)y^i$$

$z = Dy$

② CREDIT & INSURANCE: $(K=2)$

$$x_H^i = \overset{\circ}{x}{}_H^i + (1+r)y_1^i + \underbrace{d_{H2}}_{=0 \text{ IN EXAMPLE}} y_2^i$$

$$x_L^i = \overset{\circ}{x}{}_L^i + (1+r)y_1^i + d_{L2}\, y_2^i$$

EQUILIBRIUM:

DEFN: AN EQUILIBRIUM OF THE MARKETS DEFINED BY D IS A $(q; (y^i)_N (x_0^i, (x_s^i)_S)_N) \in \mathbb{R}_+^K \times \mathbb{R}^{NK} \times \mathbb{R}_+^{N(1+S)}$ THAT SATISFIES

(U-MAX) $\forall i \in N$: $(y^i, x_0^i, (x_s^i)_S)$ MAXIMIZES u^i s.t.

$$x_0^i + q\cdot y^i \leqq \overset{\circ}{x}{}_0^i$$

$z = Dy$

AND $x_s^i \leqq \overset{\circ}{x}{}_s^i + \sum_{k\in K} d_{sk}\, y_k^i$, $s = 1, \dots, S$

(M-CLR) $\sum_{i\in N} x_0^i \leqq \sum_{i\in N} \overset{\circ}{x}{}_0^i$ AND

AND $\sum_{i\in N} y_k^i \leqq 0$ AND $q_k \sum_{i\in N} y_k^i = 0$, $k = 1, \dots, K$.

→ IS THE EQUILIBRIUM ALLOCATION PARETO OPTIMAL?

IS IT THE ARROW-DEBREU ALLOCATION?

ARE PRICES q RELATED TO A-D PRICES?

THEOREM: LET $(p, (x_0^i, (x_s^i)_S)_N)$ BE AN ARROW-DEBREU EQUILIBRIUM FOR THE ECONOMY $(S, (u^i, (\mathring{x}_0^i, (\mathring{x}_s^i)_S))_N)$. LET D BE AN $S \times K$ SECURITIES RETURNS MATRIX, AND LET $q = pD$. IF $\operatorname{rank} D = S$, THEN THERE IS A $(y^i)_{i \in N}$ IN $\mathbb{R}^{NK}$ SUCH THAT $(q, (y^i)_N, (x_0^i, (x_s^i)_S)_N)$ IS AN EQUILIBRIUM OF THE SECURITIES MARKETS DEFINED BY D.

[NOTE THAT THE ALLOCATION, $(x_0^i, (x_s^i)_S)_{i \in N}$, IS THE SAME IN BOTH EQUILIBRIA.]

PROOF: IN THE PROPOSITION ON THE FOLLOWING PAGE

WE SHOW, IN THE PROPOSITION ON THE FOLLOWING PAGE, THAT EACH CONSUMER FACES THE SAME BUDGET SET IN THE CONSUMPTION-PLAN SPACE $\mathbb{R}_+^{(1+S)}$ IN BOTH MARKETS (AT THE RESPECTIVE EQUILIBRIUM PRICES); THEREFORE EACH CONSUMER WILL CHOOSE THE SAME PLAN AT ONE EQUILIBRIUM PRICE-LIST AS AT THE OTHER. ||

REMARK: THE CONDITION THAT $\operatorname{rank} D = S$ CAN BE STATED EQUIVALENTLY AS:

THE SECURITIES $d_1, \ldots, d_K$ SPAN THE SPACE $\mathbb{R}^S$.

IN EACH CASE THE CONSUMER IS CHOOSING SO AS TO MAXIMIZE $u(x_0, (x_s)_S)$, i.e., $u(\overset{\circ}{x}_0 + z_0, (\overset{\circ}{x}_s + z_s)_S)$.

ARROW-DEBREU:

CONSUMER CHOOSES $(z_0, (z_s)) \in \mathbb{R}^{1+S}$ s.t. $\quad$ $1+S$ VAR'S

B.C.: $z_0 + \sum_{s \in S} p_s z_s = 0 \leftarrow z_0 + p \cdot z = 0$ $\quad$ 1 CONSTR.

WITH SECURITIES:

CONSUMER CHOOSES $y \in \mathbb{R}^K$ AND $(z_0, (z_s)) \in \mathbb{R}^{1+S}$ $\quad$ $K+1+S$ VAR

B.C.: $z_0 + \sum_{k \in K} q_k y_k = 0 \leftarrow z_0 + q \cdot y = 0$ $\quad$ 1

AND $z = Dy$ i.e., $z_s = \sum_{k \in K} d_{sk} y_k$, $s = 1, \ldots, S$ $\quad$ S $\Big\}$ $1+S$ CONS

DEFINE

$$A := \{ \underbrace{(z_0, (z_s))}_{\in \mathbb{R}^{1+S}} \mid z_0 + p \cdot z = 0 \}$$

$$B := \{ (z_0, (z_s)) \in \mathbb{R}^{1+S} \mid \exists y \in \mathbb{R}^K : z_0 + q \cdot y = 0 \;\&\; z = Dy \}$$

PROPOSITION: LET $p \in \mathbb{R}^S$; LET D BE $S \times K$; LET $q = pD \in \mathbb{R}^K$; AND LET A AND B BE AS ABOVE. IF rank $D = S$, THEN $A = B$.

PROOF:

WE SHOW $A \subseteq B$ AND $B \subseteq A$. (NEXT PAGE)

PROOF THAT $A = B$: (i) $A \subseteq B$, (ii) $B \subseteq A$

(i) LET $(z_0, z) \in A$;

i.e., $0 = z_0 + p \cdot z$ ← $\exists y : z = Dy$ BECAUSE D HAS FULL RANK

$= z_0 + p \cdot (Dy)$ FOR SOME y

$= z_0 + (pD) \cdot y$

$= z_0 + q \cdot y$ ← BECAUSE $q = pD$.

$\therefore (z_0, z) \in B$.

(ii) LET $(z_0, z) \in B$;

i.e. $0 = z_0 + q \cdot y$ FOR SOME y s.t. $z = Dy$

$= z_0 + (pD) \cdot y$ ← BECAUSE $q = pD$

$= z_0 + p \cdot (Dy)$

$= z_0 + p \cdot z$.

$\therefore (z_0, z) \in A$. ||

WE'VE SHOWN THAT IF $q = pD$ AND IF D IS OF FULL RANK (i.e., rank $D = S$), THEN THE SET OF NET CONSUMPTION PLANS (z_0, z) ACHIEVABLE BY A CONSUMER — i.e., THE SET A UNDER ARROW-DEBREU CONTINGENT CLAIMS, AND THE SET B VIA SECURITIES MARKETS — IS THE <u>SAME</u> SET IN EACH OF THE TWO EQUILIBRIA.