BERTRAND EQUILIBRIUM WITH A HOMOGENEOUS GOOD

(SEE JEHLE & RENY, SECTION 4.2.2)

LET $Q = D(p)$ BE THE MARKET DEMAND FUNCTION. WE ASSUME THAT

$$q_1(p_1, p_2) = \begin{cases} D(p_1), & \text{IF } p_1 < p_2 \\ \frac{1}{2}D(p_1), & \text{IF } p_1 = p_2 \\ 0, & \text{IF } p_1 > p_2, \end{cases}$$

AND SIMILARLY FOR FIRM 2. IN OTHER WORDS, THE HIGH-PRICE FIRM SELLS ZERO WHILE THE LOW-PRICE FIRM SELLS THE ENTIRE MARKET DEMAND; AND THE FIRMS DIVIDE THE MARKET EQUALLY IF THEY CHARGE THE SAME PRICE. (NOTHING WILL BE CHANGED IF WE ASSUME INSTEAD THAT WHEN $p_1 = p_2 = p$ WE HAVE $q_1(p_1, p_2) + q_2(p_1, p_2) = D(p)$ AND $q_i(p_1, p_2)$ BOUNDED AWAY FROM ZERO FOR $i = 1, 2$.)

THE DISCONTINUITY IN THE FIRMS' RESIDUAL DEMAND FUNCTIONS YIELDS A COMPLETELY DIFFERENT OUTCOME AT THE BERTRAND EQUILIBRIUM THAN IN THE CASE OF DIFFERENTIATED PRODUCTS:

PROPOSITION: IF TWO FIRMS THAT PRODUCE A HOMOGENEOUS GOOD HAVE IDENTICAL LINEAR COST FUNCTIONS $C(q_i) = cq_i$, THEN THE UNIQUE BERTRAND EQUILIBRIUM SATISFIES $p_1 = p_2 = c$.

PROOF:

FIRST WE SHOW THAT $(p_1, p_2) = (c, c)$ IS A BERTRAND EQUILIBRIUM. IF $p_2 = c$, THEN

$$\pi_1(p_1, p_2) = 0 \text{ IF } p_1 \geqq c$$
$$\text{AND } \pi_1(p_1, p_2) < 0 \text{ IF } p_1 < c.$$

THEREFORE $p_1 = c$ MAXIMIZES $\pi_1(p_1, p_2)$ (ALTHOUGH NOT UNIQUELY). SIMILARLY, IF $p_1 = c$ THEN $p_2 = c$ MAXIMIZES $\pi_2(p_1, p_2)$. THEREFORE $(p_1, p_2) = (c, c)$ IS A BERTRAND EQUILIBRIUM.

NOW WE SHOW THAT THERE IS NO OTHER BERTRAND EQUILIBRIUM:

SUPPOSE FIRST THAT $p_i < c$ FOR AT LEAST ONE FIRM i. THEN THE LOW-PRICE FIRM (AND BOTH FIRMS IF $p_1 = p_2$) SELLS A POSITIVE QUANTITY AND ITS PROFIT IS NEGATIVE, SO IT IS NOT MAXIMIZING PROFIT ($\pi_i = 0$ IF $p_i = c$). THEREFORE THERE IS NO BERTRAND EQUILIBRIUM WITH $p_i < c$ FOR EITHER $i = 1$ OR $i = 2$.

NOW SUPPOSE THAT $p_2 = c$ AND $p_1 > p_2$; THEN

$$\pi_2(p_1, p_2) = 0 \quad (\text{BECAUSE } p_2 = c)$$
$$\text{AND } \pi_2(p_1, p_1) > 0,$$

SO π_2 IS NOT MAXIMIZED AT $p_2 = c$, AND THIS IS NOT A BERTRAND EQUILIBRIUM. SIMILARLY, IT IS NOT A BERTRAND EQUILIBRIUM IF $p_1 = c$ AND $p_2 > p_1$.

THIS LEAVES THE CASE $p_1, p_2 > c$. SUPPOSE $p_1 > p_2 > c$; THEN

$$\pi_1(p_1, p_2) = 0,$$
$$\text{BUT } \pi_1(p_1, p_2) > 0 \text{ IF } p_1 = p_2,$$

$p_2 > p_1 > c$

SO p_1 DOES NOT MAXIMIZE π_1, AND THIS IS NOT A BERTRAND EQUILIBRIUM. SIMILARLY, A BERTRAND EQUILIBRIUM DOES NOT HAVE

FINALLY, SUPPOSE $p_1 = p_2 > c$: THEN WE HAVE

$$\pi_1(p_1, p_2) = \tfrac{1}{2}(p_1 - c)D(p_1) = \tfrac{1}{2}(p_2 - c)D(p_2).$$

BUT IF $p_1 = p_2 - \delta$, THEN

$$\pi_1(p_1, p_2) = (p_2 - \delta - c)D(p_2 - \delta);$$

SEE THE DIAGRAMS ON THE NEXT PAGE

THEREFORE $\lim_{\delta \to 0} \pi_1(p_2 - \delta, p_2) = (p_2 - c)D(p_2)$. [ASSUMING $D(\cdot)$ IS CONTINUOUS]

THUS, FOR δ SUFFICIENTLY SMALL, WE HAVE

$$\pi_1(p_2 - \delta, p_2) > \pi_1(p_2, p_2) = \tfrac{1}{2}(p_2 - c)D(p_2),$$

~~SO π_1 IS NOT MAXIMIZED AT $p_1 = p_2$, AND THEREFORE THERE IS NO BERTRAND EQUILIBRIUM AT WHICH $p_1 = p_2 > c$.~~ ||

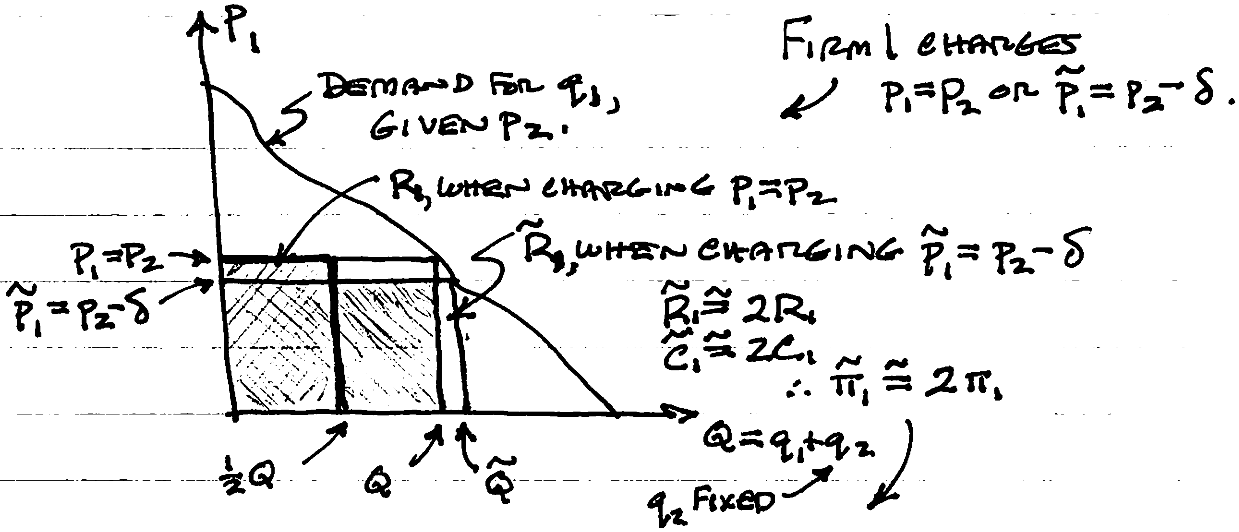
REMARK: IF THE SET OF POSSIBLE PRICES IS DISCRETE, THEN THE UNIQUE BERTRAND EQUILIBRIUM SATISFIES $p_1 = p_2 = p'$, WHERE p' IS THE SMALLEST PRICE THAT SATISFIES $p' > c$.

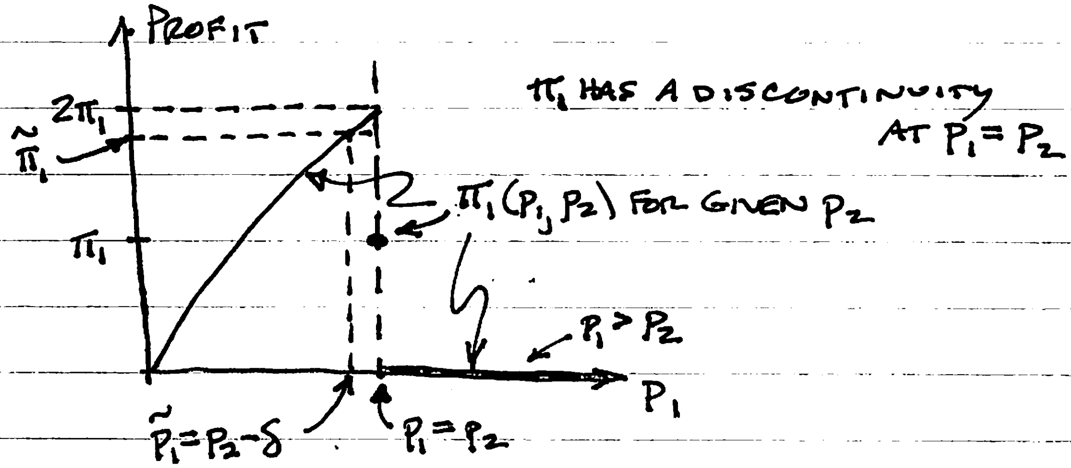
REMARK: IF THE NUMBER OF FIRMS IS GREATER THAN TWO (ALL WITH THE SAME LINEAR COST FUNCTION), THERE ARE MULTIPLE BERTRAND EQUILIBRIA, IN EACH OF WHICH THE MARKET PRICE IS $p = c$.

THE REMARKS CAN BE PROVED VIA ARGUMENTS THAT ARE MUCH THE SAME AS IN THE PROOF ABOVE.

THE CASE $p_1 = p_2 > c$:

FIRM 1 IS TAKING p_2 AS GIVEN, AND CAN INCREASE π_1 BY CHOOSING $\tilde{p}_1 = p_2 - \delta$:





NOTE THAT THE ARGUMENT DEPENDS CRITICALLY ON (1) THE FIRMS HAVING THE SAME COST FUNCTION, AND (2) CONSTANT RETURNS TO SCALE — i.e., MARGINAL COST REMAINING CONSTANT, SO THAT $C(2q_i) = 2C(q_i)$.