Lecture Notes

on

Public Goods, Externalities, and Mechanism Design

Economics 501B

University of Arizona

Revised 11/30/2017

Introductory Notes on Public Goods
for Intermediate Microeconomics

Let's begin with an extremely simple example of a public good. Suppose there are only two people who live on the shore of Lake Magnavista. Amy likes to water ski and Bev likes to sunbathe. Both activities are seriously affected by the level of the water in the lake. When there is a lot of water in the lake, it's good for water skiing but the water line is so high that there is no beach for sunbathing. When there is much less water, the sunbathing is good but the lake is too shallow for water skiing. Therefore Amy prefers that the lake have lots of water, and Bev prefers that it have much less water. Fortunately, it's possible to raise or lower the water level costlessly, by opening a dam at one end of the lake or at the other end. Unfortunately, it's not clear at what level the water ought to be set.

In order to have a measure of the amount of water in the lake, let's use the water's depth at a specified location on the lake: let x denote the water's depth (in feet) at that location. Amy's and Bev's preferences are described by the following utility functions:

$$u^A(x, y_A) = y_A - (15 - x)^2 \quad \text{and} \quad u^B(x, y_B) = y_B - \frac{1}{2}(6 - x)^2,$$

where x denotes the water level and y_A and y_B are Amy's and Bev's daily consumption of other goods, measured in dollars. Suppose Amy and Bev each have incomes of \$100 per day. Note that Amy's and Bev's marginal rates of substitution are

$$MRS_A = 30 - 2x \quad \text{and} \quad MRS_B = 6 - x.$$

Amy's most-preferred water level is $\widehat{x}_A = 15$ and Bev's most-preferred level is $\widehat{x}_B = 6$. Figure 1 depicts their indifference maps. Notice that if the water level is above $\widehat{x}_B$, Bev would be willing to pay (*i.e.*, to give up some of the y-good) to have x *reduced*, and that Amy would similarly be willing to pay to reduce x if it's above her ideal level, $\widehat{x}_A$.

What makes this situation different than everything we've seen before is that the x variable can't be at different levels for different people. It's not like pizza or beer, where Amy can consume one quantity and Bev a different quantity. In this case, the water level can be varied, but it will be the same for both of them. That's why we haven't used A and B subscripts on the x variable: it's just one variable, not two. The water level in this example is a **public good**.

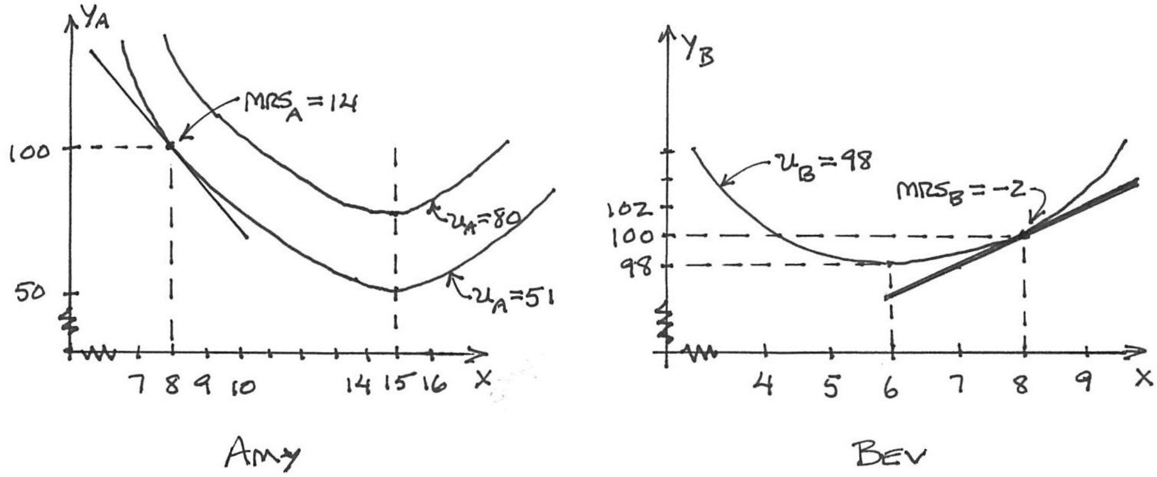


Figure 1

Let's try to determine which outcomes are Pareto efficient. Let's start by asking whether a water level of $x = 8$ feet is efficient. Figure 2 will be helpful here: it's a diagram similar to the Edgeworth Box. The difference is that here in this diagram the "corners" or "origins" for both consumers are placed on the left edge of the box, so that when x increases (*i.e.*, as we move toward the right), *both* persons' consumption of the x-good is increasing. This is in contrast to the Edgeworth Box, where as we move toward the right, Person A's consumption is increasing and B's is decreasing, because with "private goods" like pizza or beer, the more one person gets, the less is available for the other person.

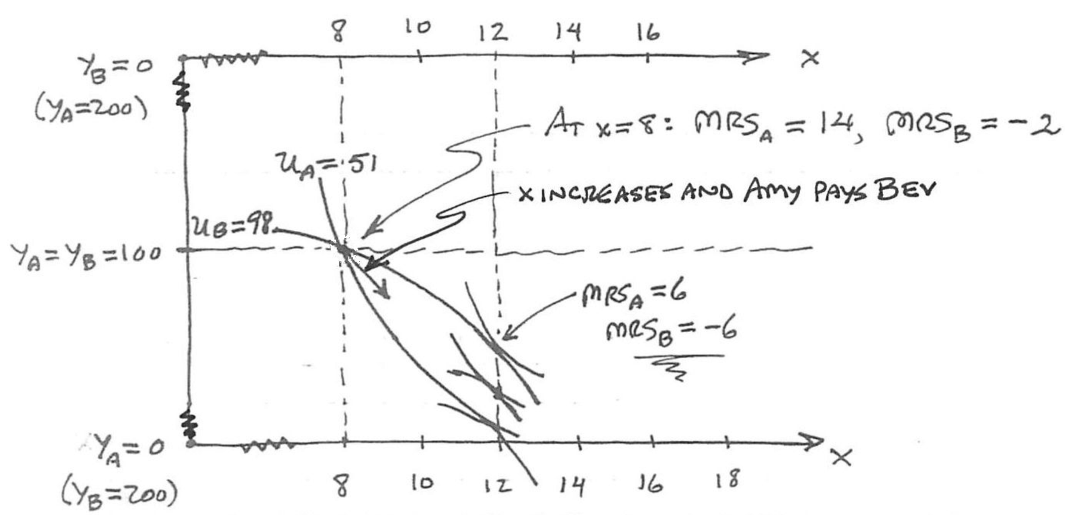


Figure 2

At $x = 8$, Amy's and Bev's MRS's will be $MRS_A = 14$ and $MRS_B = -2$. Amy would be willing to pay \$14 to increase the water level one foot and Bev would be willing to pay \$2 to decrease it by one foot — or Bev would be willing to accept a payment of \$2 as compensation for *increasing* the water level by one foot. Therefore, if we were to increase the water level by a foot, and if Amy were to *compensate* Bev by paying her, say, \$6, then they would both be better off. In fact, you can calculate that Amy's utility will increase from 51 to 58 and that Bev's utility will increase from 98 to $101\frac{1}{2}$.

So what about this new water level of $x = 9$ feet — is *it* efficient? We have $MRS_A = 12$ and $MRS_B = -3$, so we could increase the water level by another foot, with Amy paying another \$6 to compensate Bev. The \$6 payment is less than the \$12 Amy would have been willing to pay, and more than the \$3 Bev would have been willing to accept as compensation, so they're again both better off from the one-foot increase with \$6 compensation. You can calculate that their utilities will have increased again, to $u_A = 63$ and $u_B = 104$.

Now it's becoming clear that as long as Amy would be willing to pay more for an increase than Bev would be willing to accept as compensation, then such a bargain — increasing the water level, with Amy compensating Bev — will make them both better off. In other words, the water level is not Pareto efficient so long as $MRS_A > -MRS_B$ — *i.e.*, so long as $MRS_A + MRS_B > 0$. When $MRS_A + MRS_B > 0$ the **marginal social value** of an increase in x is positive, so x should be increased. Similarly, we could show that if $MRS_A + MRS_B < 0$, then x should be decreased (because the marginal social value of an increase is negative, so the marginal social value of a *decrease* in x is positive).

The Pareto efficient outcomes are therefore the ones that satisfy the Test Condition $MRS_A + MRS_B = 0$. In our example, it's easy to solve for the efficient water level in the lake:

$$MRS_A + MRS_B = (30 - 2x) + (6 - x) = 36 - 3x;$$

therefore the Test Condition yields

$$36 - 3x = 0 \quad i.e., \quad x = 12.$$

The efficient water level is 12 feet, where $MRS_A = 6$ and $MRS_B = -6$.

Now we know the water level that's Pareto efficient. But what level will actually be chosen? As usual, that depends on the institutional arrangements that are used for choosing the water level. For example, the affected parties might vote on the level they want. The outcome of that institution can be analyzed using game theory, which we won't do here. Let's suppose instead that Bev owns the lake, or at least that she has the right to choose the water level. What will the water level be? It seems that Bev will choose the level that she likes best, namely $x = 6$ feet.

But we've already seen that at such a low water level, Bev would be better off to allow a higher level if Amy will compensate her appropriately. Indeed, just exactly as in our discussion earlier in the semester of bargaining between two parties, we would expect them to arrive at a mutually agreeable bargain in which there are no more gains to be had from further trade or bargaining — *i.e.*, at a Pareto efficient outcome. In our example this means that the water level will be 12 feet, with Amy paying Bev some amount as compensation for the increase from 6 to 12 feet. In Figure 3 they will end up on the vertical line at $x = 12$, with Amy paying Bev an amount that leaves their y consumptions between their initial indifference curves, the ones that pass through the allocation in which $x = 6$ and $y_A = y_B = 100$.

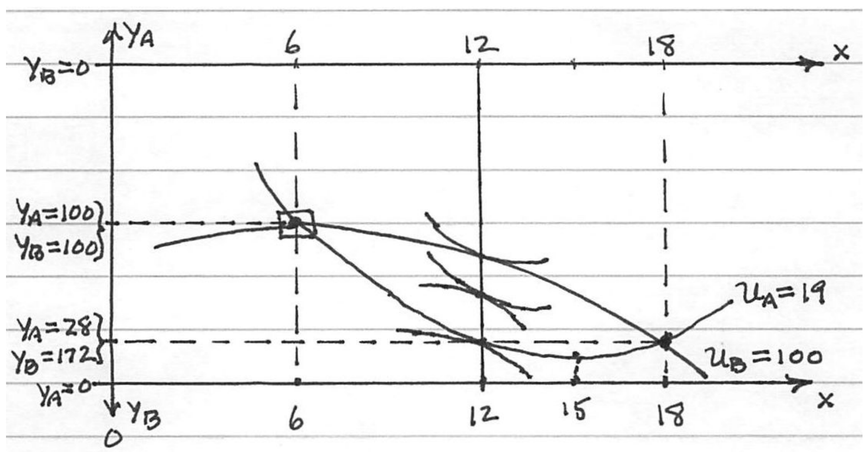


Figure 3

What if instead Amy has the right to choose the water level? At first it seems as if she would choose the level *she* likes best, $x = 15$ feet. But the same argument as in the preceding paragraph tells us that again we should expect the two women to bargain with one another to set the water level at $x = 12$ feet, but this time with Bev paying some compensation to Amy for setting the water level below what Amy would like the most.

It was the economist Ronald Coase who pointed out that the assignment of legal rights will affect only who pays compensation, and how much, but will not affect the level of the activity that affects both parties. He was awarded the Nobel Prize for this idea, which has had a profound effect on law and, to a lesser extent, legislation in recent decades.

What if the public good is not costless, as the water level was? Then the Pareto Efficiency Test Condition (PETC) is that the marginal social value of a one-unit increase in the amount of the public good must be equal to the marginal cost of producing that additional unit — *i.e.*, that the sum of all the MRS's must be equal to MC:

$$(PETC) \qquad MRS_1 + MRS_2 + \cdots + MRS_n = MC.$$

Notice that in our lake example the MC of changing the water level was zero, so this Test Condition is actually the same as the one we used in the example, just generalized to account for costly public goods as well as costless ones.

Another example: Suppose Amy and Bev are plagued by mosquitoes, but it's possible to control the number of mosquitoes by spraying regularly. However, the mosquito spray can't be confined to the property of just one of the women: any spray that's applied affects the entire lake shore equally. Suppose Amy's and Bev's preferences are described by the same utility functions as in our water-level example, where x now denotes the number of gallons that are sprayed each week, and y_A and y_B still denote the amounts Amy and Bev spend on other goods. If each gallon of spray were free, our problem would be exactly the same as before, because we would have $MC = 0$. But suppose instead that the spray costs \$24 per gallon. Then our Test Condition (PETC) yields the following:

$$MRS_A + MRS_B = MC : \quad (30 - 2x) + (6 - x) = 24; \qquad i.e.,\ 36 - 3x = 24.$$

Therefore the Pareto efficient amount of mosquito spray is 4 gallons per week, which equates the marginal social value to the marginal social cost, \$24.

How much will be sprayed if Amy and Bev each choose on the basis of just their own personal benefits from the spray? Bev won't choose to spray at all, because even when $x = 0$ her MRS is much less than the price of the spray (\$6 vs. \$24). Amy will choose $x = 3$ gallons of spray, which equates her own MRS to the \$24 price of the spray. So the amount of spray that's chosen will be less than the Pareto efficient amount. Both women would be better off if an additional gallon were sprayed and Amy were to pay, say, \$22 of the cost of the additional gallon and Bev paid the remaining \$2.

A more striking example: Suppose there are 100 homeowners living on the lake shore, and that each one has $MRS = 6 - x$, where x is the number of tankfuls that are sprayed. Suppose that the cost of the mosquito spray is \$100 per tankful. Then the Pareto efficient amount of spray is $x = 5$ tankfuls:

$$\sum MRS_i = MC : \quad 100(6 - x) = 100; \qquad i.e.,\ 600 - 100x = 100.$$

How much will actually be sprayed if each homeowner chooses on the basis of just his own benefit from the spray? At $x = 0$ each homeowner has $MRS = 6$ — *i.e.*, each would be willing to pay \$6 to increase x from $x = 0$ to $x = 1$ tankful sprayed. But the cost of such an increase — \$100 — far exceeds each person's MRS. Therefore no one will choose to spray. Even though the marginal social benefit is \$600 and the marginal cost is only \$100, no spray is forthcoming.

Public Goods: Pareto Efficiency and Market Outcomes

A Motivating Example: Water Skiing vs. Sunbathing

Read the Introductory Notes for Microeconomics. Note that the Pareto marginal condition for two persons is shown to be *not* $MRS^1 = MRS^2 = MC$, but $MRS^1 + MRS^2 = MC$ instead. And since $MC = 0$ in the example, we obtained $MRS^1 + MRS^2 = 0$.

A Second Motivating Example: Mosquito Spray

The homeowners in a residential neighborhood are plagued by mosquitoes. The number of mosquitoes can be controlled by spraying. The mosquito spray is a public good because whatever amount is sprayed, this is the amount that is experienced (for good or bad) by all the homeowners: it's not possible to contain the spray so as to affect only the homeowner who purchases it. Let x denote the number of tankfuls of spray that are sprayed. For each $i \in N = \{1, \ldots, n\}$ let y_i denote household i's dollar expenditure on other goods, and let $u^i(x, y_i)$ be household i's utility function. An allocation is an $(n+1)$-tuple $(x, y_1, \ldots, y_n)$.

Pareto Efficiency:

We first derive the marginal conditions that characterize the Pareto allocations. The Pareto maximization problem is

$$\begin{aligned} \max_{x,(y^i)_1^n} \lambda_1 u^1(x, y_1) \quad & \text{subject to } x, y_1, \ldots, y_n \geqq 0 \\ \sum_{i=1}^n y_i + C(x) \ \leqq \ & \mathring{y}, \qquad\qquad (\sigma) \\ u^i(x, y_i) \ \geqq \ & u_i, \quad i = 2, ..., n. \qquad (\lambda_i) \end{aligned} \tag{P-Max}$$

The first-order marginal conditions for an interior solution are

$\exists\, \sigma \geqq 0$ and $\lambda_2, ..., \lambda_n \geqq 0$ such that

$$\lambda_1 u_x^1 + \lambda_2 u_x^2 + \ldots \lambda_n u_x^n = \sigma C'(x) \quad \text{and} \quad \lambda_i u_y^i = \sigma, \quad i = 1, ..., n. \tag{FOMC}$$

Combining these first-order equations yields

$$\sum_{i=1}^n MRS^i = MC,$$

which is the **Samuelson Marginal Condition** for Pareto efficiency with a public good.

The Market Outcome:

In our mosquito-spray example, assume that there's a market in which firms provide mosquito-control spraying service at a price p per tankful of spray. Let's also assume that the homeowners are price-takers. In this case that's not enough to define the individual homeowner's decision problem: the amount of spray an individual wishes to purchase will be affected by how much the other homeowners purchase. What the individual homeowner cares about is the total amount of spray purchased by *everyone*, which we've denoted by x. Let ξ_i denote the amount of spray purchased by individual i; and let X_{-i} denote the total purchased by everyone else: $X_{-i} = \sum_{j \neq i} \xi_j$. Then $x = \sum_1^n \xi_j = X_{-i} + \xi_i$. Let's assume, then, that each individual i takes both the market price p *and* the total amount purchased by all the others, X_{-i}, as given.

The decision problem for each individual i is

(U-max) $$\max_{(\xi_i, y_i) \in \mathbb{R}^2_+} u^i(x, y_i) = u^i(X_{-i} + \xi_i, y_i) \text{ subject to } p\xi_i + y_i \leqq \mathring{y}_i,$$

or equivalently,

$$\max\ u^i(X_{-i} + \xi_i, \mathring{y}_i - p\xi_i) \text{ for } \xi_i \in [0, \mathring{y}_i/p].$$

The first-order marginal condition (assuming that $\xi_i < \mathring{y}_i/p$) is

$$MRS^i \leqq p \qquad \text{and} \qquad MRS^i = p \text{ if } \xi_i > 0.$$

The diagrams in Figure 1 depict the individual's decision problem. The total amount everyone else has purchased is X_{-i}. That's the smallest level of x individual i can obtain, by choosing $\xi_i = 0$. And it's the level he *will* choose unless his MRS^i at X_{-i} exceeds p. If his MRS^i does exceed p at X_{-i}, he will choose ξ_i (and therefore x) up to the level at which his $MRS^i = p$.

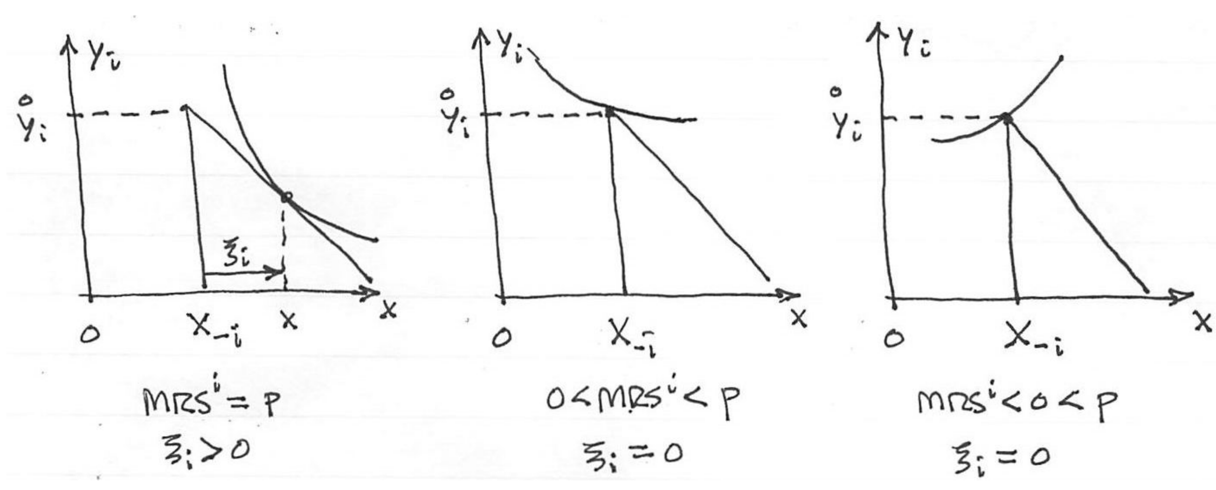


Figure 1

This leads naturally to the following definition of equilibrium:

Definition: Let p be the price at which a public good is provided. A **public-good price-taking Nash equilibrium** at price p is an n-tuple $(\xi_1, \ldots, \xi_n) \in \mathbb{R}^n_+$ that satisfies (U-max) for each $i = 1, \ldots, n$.

Clearly, if an equilibrium has $x > 0$, then some individual $h \in N$ must satisfy $\xi_h > 0$ and therefore $MRS^h = p$. Each i whose MRS^i is less than p will not purchase any of the public good (*i.e.*, $\xi_i = 0$ for each such i), but some or all of these individuals' marginal rates of substitution — their marginal values for the public good — may nevertheless be well above zero. Consequently we would have $\sum_1^n MRS^i > p$, and indeed the sum will often be substantially larger than p.

For the n consumers of the public good, note that the marginal cost *to them* of an additional unit of the good is its price p. Thus, to the n consumers, a market equilibrium typically satisfies the inequality $\sum_1^n MRS^i > MC$ — the equilibrium is not Pareto efficient, because the equilibrium level of x is too low. And if $\sum_1^n MRS^i$ is *substantially* larger than p, then the equilibrium x may be substantially less than Pareto efficiency would require, as in the following examples.

Example 1: There are five homeowners: $N = \{1, 2, 3, 4, 5\}$. Their utility functions are all of the form $u(x, y_i) = y_i - \frac{1}{2}(\alpha_i - x)^2$, where x denotes the level at which a public good is provided, and y_i denotes the amount of money homeowner i has available to spend on other goods. The values of their preference parameters α_i are

$$\alpha_1 = 30, \qquad \alpha_2 = 27, \qquad \alpha_3 = 24, \qquad \alpha_4 = 21, \qquad \alpha_5 = 18,$$

and their MRS functions are therefore

$$MRS^1 = 30 - x, \quad MRS^2 = 27 - x, \quad MRS^3 = 24 - x, \quad MRS^4 = 21 - x, \quad MRS^5 = 18 - x.$$

The firms that produce the public good all charge a per-unit price of p dollars; p is therefore the marginal cost to the homeowners for each unit of x. Suppose $p = \$40$. Because the utility functions are quasilinear, there's a unique Pareto level of the public good, namely $x = 16$:

$$\Sigma MRS^i = 120 - 5x \text{ and } MC = 40, \quad \text{therefore } \Sigma MRS^i = MC \text{ at } x = 16.$$

And because each $i \in N$ has $MRS^i < p$ at every $x \in \mathbb{R}_+$, there is also a unique equilibrium:

$$\xi_i = 0 \text{ for all } i, \quad \text{and therefore } x = 0.$$

None of the public good is purchased, despite the fact that the Pareto level is $x = 16$ and despite the fact that when $x = 0$, the marginal social value of the public good, ΣMRS^i, is 120, which far exceeds the marginal cost (*i.e.*, the \$40 price) of each unit of x. Consumer surplus at the Pareto provision level, $x = 16$, is \$640, all of which is foregone at the equilibrium.

Example 2: With the same five consumers as in Example 1, suppose $p = \$20$. In this case the unique Pareto level of x is $x = 20$. There is again a unique equilibrium: $(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5) = (10, 0, 0, 0, 0)$ and therefore $x = 10$. Note that at the equilibrium outcome, $MRS^1 = \$20 = p$ and for all other i, $0 < MRS^i < p$, as in the middle diagram in Figure 1. In everyday, nontechnical language, the other four consumers would be said to be "free riding" on Consumer 1, and would be referred to as "free riders:" they're purchasing none of the public good while receiving positive benefit from Consumer 1's purchase. We have $\Sigma MRS^i = 70$, which is substantially larger than the \$20 price. Consumer surplus at the Pareto provision level, $x = 20$, is \$1000; consumer surplus at the equilibrium is \$750.

Example 3: Suppose the price is $p = \$20$ as in Example 2, but that the preference parameters in Examples 1 and 2 are changed to

$$\alpha_1 = \alpha_2 = \alpha_3 = 30, \qquad \alpha_4 = 25, \qquad \alpha_5 = 5,$$

so that the MRS functions are now

$$MRS^1 = MRS^2 = MRS^3 = 30 - x, \quad MRS^4 = 25 - x, \quad MRS^5 = 5 - x.$$

We still have $\Sigma MRS^i = 120 - 5x$, so the Pareto level of x is still $x = 20$, as in Example 2. But now there are multiple equilibria: the equilibria are all the $(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5)$ that satisfy $\xi_1 + \xi_2 + \xi_3 = 10$ and $\xi_4 = \xi_5 = 0$. In each of the equilibria we have $MRS^i = p = 20$ for $i = 1, 2, 3$, and we have $MRS^4 = 15 < p$ and $MRS^5 = -5 < p$. Note that Consumer 5 is not a free rider here: her consumer surplus is zero. She receives some surplus on the first five units of x, which is just offset by the negative consumer surplus she receives from the next five units. Her decision problem corresponds to the rightmost diagram in Figure 1.

Example 4: In Example 3, change α_4 to 28 and α_5 to 2. The Pareto level of x and the equilibria are unchanged, but now $MRS^4 = 18$ and $MRS^5 = -8$ at the equilibria. Now Consumer 5 "suffers damages" at the equilibrium: her consumer surplus is $-\$30$. She is worse off than if none of the public good were provided. For example, think of a homeowner who experiences respiratory difficulties from mosquito spray if it's provided at a level greater than $x = 2$.

An Alternative Institution:

Suppose the homeowners form a homeowners association (HOA) to deal with their mosquito problem: the HOA will accept voluntary contributions from the homeowners, and will then use the total contributions to purchase as much mosquito spray as the contributions will buy. More formally, each homeowner $i \in N$ chooses to contribute a dollar amount $t_i \in \mathbb{R}_+$ to the mosquito fund. The total amount contributed is $\Sigma_{i \in N} t_i$. Then the HOA purchases $x = \frac{1}{p}\Sigma_{i \in N} t_i$ tanks of spray, where p is the price per tank.

We'll assume that each homeowner takes the total of all the *others'* contributions as given, and chooses his own contribution t_i to maximize his utility. Let T_{-i} denote the total of the others' contributions: $T_{-i} = \Sigma_{j \neq i} t_j$. The individual's maximization problem is

$$(*) \qquad \max_{t_i \in \mathbb{R}_+} u^i(x, y_i), \quad \text{where } x = \frac{1}{p}(T_{-i} + t_i) \text{ and } y_i = \mathring{y} - t_i.$$

We define an equilibrium as follows:

Definition: A **voluntary contributions equilibrium** for a public good with price $p \in \mathbb{R}_{++}$ is an n-tuple $(t_1, \ldots, t_n) \in \mathbb{R}^n_+$ in which, for each $i \in N$, t_i is a solution of $(*)$.

The first-order marginal condition for each individual's maximization problem $(*)$ is

$$\frac{1}{p}u^i_x - u^i_y \leqq 0 \ \text{ and } \ \frac{1}{p}u^i_x - u^i_y = 0 \text{ if } t_i > 0,$$

i.e.,

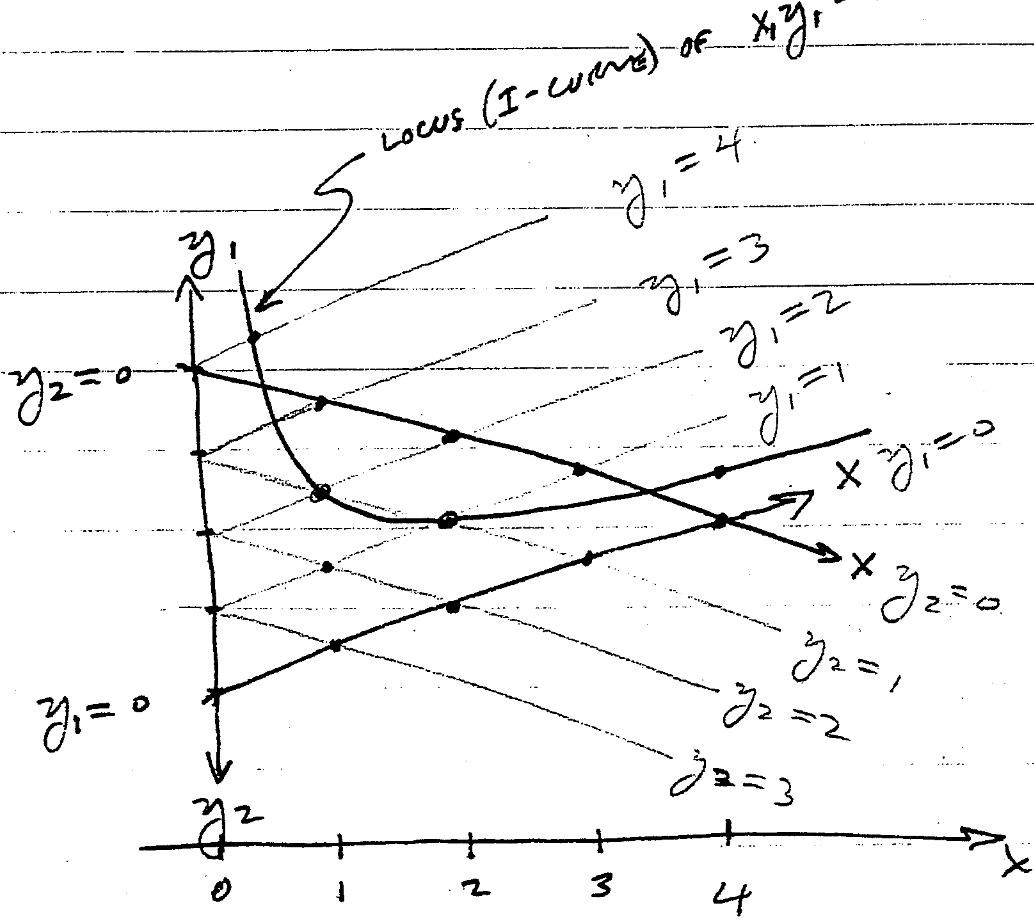
$$MRS^i \leqq p \ \text{ and } \ MRS^i = p \text{ if } t_i > 0.$$

This is the same marginal condition as in the individual-purchases institution we analyzed above. Therefore a voluntary contributions equilibrium and a price-taking equilibrium are identical, with the individual actions t_i and ξ_i related by the equations $t_i = p\xi_i$ for each $i \in N$. In Example 1 we have $t_i = 0$ for each $i \in N$, and $x = 0$. In Example 2 we have $t_1 = \$200$ and $t_i = 0$ for $i = 2, 3, 4, 5$, and $x = 10$. In Examples 3 and 4 we have $t_1 + t_2 + t_3 = \$200$ and $t_4 = t_5 = 0$, and $x = 10$.

THE KOLM TRIANGLE

(AFTER SERGE-CHRISTOPHE KOLM)

THE COSTLY-PUBLIC-GOOD ANALOGUE OF THE "EDGEWORTH STRIP" IN THE COSTLESS CASE AND THE EDGEWORTH BOX IN THE NO-EXTERNALITIES CASE.



$$\overset{\circ}{y}_1 + \overset{\circ}{y}_2 = 4$$

$$C(x) = x \quad (MC = 1).$$

$$\therefore \; y_1 + y_2 = 4 - x.$$

Public Goods: Examples

The classical definition of a public good is one that is **non-excludable** and **non-rivalrous.** The classic example of a public good is a **lighthouse**. A lighthouse is:

Non-excludable because it's not possible to exclude some ships from enjoying the benefits of the lighthouse (for example, excluding ships that haven't paid anything toward the cost of the lighthouse) while at the same time providing the benefits to other ships; and

Non-rivalrous because if the lighthouse's benefits are already being provided to some ships, it costs nothing for additional ships to enjoy the benefits as well. This is not like a "rivalrous good," where providing a greater amount of the good to someone requires either that more of the good be produced or else that less of it be provided to others – *i.e.*, where there is a very real opportunity cost of providing more of the good to some people.

Some other examples of public goods:

Radio and television: Today no one who broadcasts a radio or TV program "over the air" excludes anyone from receiving the broadcast, and the cost of the broadcast is unaffected by the number of people who actually tune in to receive it (it's non-rivalrous). In the early decades of broadcasting, exclusion was not technologically possible; but technology to "scramble" and de-scramble TV signals was invented so that broadcasters could charge a fee and exclude non-payers. Scrambling technology has been superseded by cable and satellite transmission, in which exclusion is possible. But while it's now technologically possible to produce a TV or radio signal from which non-payers are excluded (so that it's not a public good), it's important to note that because TV and radio signals are non-rivalrous, they are *technologically* public goods: it's *technologically possible* to provide them without exclusion.

Clean air; pollution abatement: The quality of the air we breathe in a particular neighborhood is both non-excludable and non-rivalrous. Pollution abatement is therefore a public good.

The examples above, together with our in-class examples of the **water level in a lake** and a **mosquito-abatement program**, make it clear that the essential characteristic of a public good is that it's not possible to vary the level of the good (*i.e.*, its quantity, its quality, etc.) across people. Changing the level for one changes the level for all. In other words, it's non-rivalrous. If a non-rivalrous good is inherently non-excludable – if exclusion is not possible, as with the lake water level or with TV in the old days – then what we have is a public good. But if exclusion is technologically *possible* for a non-rivalrous good, as with TV today, then the good is *potentially* a public good, but it may not be manifested as a public good in practice.

A highway system; a mass-transit system; a school system: In each of these examples, everyone who has access to the system benefits (or suffers) from the *same* system. The system's characteristics – for example, the quality of the teaching in a school, or the time it takes the subway to travel from Brooklyn to the Bronx, or the number of potholes in the streets – these are the same for all users. However, in each of these cases it's at least *possible* to exclude people from access, and therefore to charge a fee or a toll for access. And these public goods are also all subject to *congestion* when too many people use them, so that the quality of the good may be affected by adding more users. Additional examples of public goods that are subject to congestion are **a bridge, a public swimming pool,** and **an airport.**

The characteristics of a home: The size of the TV, the color of the living room, the size of the swimming pool and the temperature of the pool's water, the size of the refrigerator, the speed of the internet connection – these are all public goods to the residents of the home. Whatever is chosen for one is chosen for all.

The thermostat setting in a room: The temperature in our seminar room, 401KK – or in a shared office, or in a home – is the same temperature for all who occupy the room. We can't make it warm for some and cool for others.

Police protection; national defense: National defense can't be provided at different levels to different citizens. Similarly for the level of police protection in a community, but police protection (unlike national defense) is subject to congestion: a larger population requires a larger police force to provide the same level of protection.

Books: Consider the information in a book. After the information has been produced (*i.e.*, once the book has been written), it needn't be written again in order for additional people to read it. So in this respect, a book is non-rivalrous. But until recently there was still a significant cost to making the book available to additional readers: additional copies of the book had to be printed, bound, and delivered, and all these steps required resources (a marginal cost) for each additional copy of the book. So the public-good character of a book (the non-rivalrous information it contained, *i.e.*, the writing) was outweighed by its private-good character, the marginal cost of producing each additional copy of the physical book. But this has completely changed: today it's possible, once a book has been written, to distribute unlimited copies electronically at essentially zero marginal cost. So today a book is a public good – *i.e.*, like a TV broadcast, a book is *technologically* a public good, but to the extent that exclusion is possible, in practice the book may or may not be a public good in any particular case.

Information; "content": Everything we said in the preceding paragraph about books is equally true for recorded music, movies, newspapers, magazines – for what's today often called "content." In the not-very-distant past, it was costly to manufacture and distribute additional copies of any particular content, such as a CD or videotape or magazine. But today, once the fixed cost of creating the content has been incurred, the marginal cost of making that content (information) available to additional people is essentially zero.

Some decisions by a corporation or a partnership: Alternative strategies or investments by a firm typically have different time-paths for their costs and returns, and often have different state-contingent outcomes. If the partners or shareholders have different time preferences or different risk preferences, the firm's decision can't generally yield outcomes that differ for the various owners – the outcomes are necessarily the same for all the owners, so the decision is a public good from the perspective of the owners.

Lindahl Equilibrium

Suppose that five homeowners live on the shore of Lake Magnavista: Amy, Bev, Cat, Dee, and Eve. In order to deal with such public goods problems as deciding on the water level in the lake and how to control mosquitoes in the summer, they've formed a homeowners' association (HOA for short).

Concerning the mosquitoes, the five women's preferences are all described by utility functions of the form $u(x, y_i) = y_i - \frac{1}{2}(\alpha_i - x)^2$, where x denotes the number of tankfuls of mosquito spray that are sprayed each week during the summer, and y_i denotes the amount of money homeowner i has available to spend on other goods. The values of their preference parameters α_i are

$$\alpha_A = 30, \qquad \alpha_B = 27, \qquad \alpha_C = 24, \qquad \alpha_D = 21, \qquad \alpha_E = 18,$$

and their MRS functions are therefore

$$MRS_A = 30 - x, \quad MRS_B = 27 - x, \quad MRS_C = 24 - x, \quad MRS_D = 21 - x, \quad MRS_E = 18 - x.$$

There are several local firms that will spray to control mosquitoes. The firms all charge the same price, $p = \$40$ per tankful they spray. This \$40 is therefore the marginal cost to the homeowners of a tankful of bug spray. Because of the homeowners' quasilinear utility functions, there is a unique Pareto amount of bug spray for them, namely $x = 16$ tanks: $\Sigma MRS_i = 120 - 5x$ and $MC = 40$, so $\Sigma MRS_i = MC$ at $x = 16$.

If the homeowners each contract separately with bug-spray firms to spray, each taking the others' purchases as given, then none of them will purchase any spray at all: the \$40 cost for each unit exceeds everyone's MRS. And as we've seen, if the HOA instead creates a fund into which they all voluntarily contribute, and uses the contributed funds to purchase the mosquito spray, the same outcome will occur: no contributions will be forthcoming, and therefore no spray will be purchased.

Now let's note that the homeowners' marginal rates of substitution *at the Pareto amount of spray* would be

$$MRS_A = 14, \qquad MRS_B = 11, \qquad MRS_C = 8, \qquad MRS_D = 5, \qquad MRS_E = 2.$$

Suppose the homeowners decide that instead of each of them purchasing bug spray separately and each paying \$40 per tankful (resulting in no spray at all being purchased), their HOA will instead charge each of them only a *share* of the \$40 price: homeowner i will pay the price-share (or per-unit tax) p_i for each unit the HOA purchases, with $\sum_{i=1}^{5} p_i = 40$.

Suppose the HOA sets these price-shares in such a way that each person's share p_i is equal to her marginal value for bug spray — *i.e.*, her MRS — at the Pareto amount $x = 16$. Then

$$p_A = 14, \qquad p_B = 11, \qquad p_C = 8, \qquad p_D = 5, \qquad p_E = 2.$$

Now the HOA asks each homeowner "How much spray *in total* do you want the HOA to purchase, knowing you will pay your price-share p_i for each tankful that's sprayed?" If each homeowner behaves as a price-taker — taking her price-share p_i as given — how much spray will she request? Choosing (x, y_i) to maximize her utility subject to the budget constraint $p_i x + y_i = \mathring{y}_i$, each homeowner will choose the x at which $MRS_i = p_i$ — *i.e.*, each homeowner will request $x = 16$.

How much money will the HOA have available to pay for the 16 tanks of spray? It will collect $p_A + p_B + p_C + p_D + p_E = \$40$ for each tank that's sprayed — *i.e.*, exactly the marginal cost to the HOA of the spray. An alternative approach would be for the HOA to draw up an agreement with one of the firms, say Bug Spray, Inc. (BSI), as follows: BSI will charge different prices p_i to each homeowner and ask each homeowner to report how much spray, in total, she would like BSI to spray at that price; and BSI is to adjust the personal prices p_i until all the homeowners are in agreement — *i.e.*, until each homeowner requests the same amount of spray. This has all the earmarks of an *equilibrium*: personal prices and the amount produced and consumed are adjusted as long as the participants don't agree on that amount; and when the participants *do* agree, adjustments no longer occur.

This idea is due to the Swedish economist Erik Lindahl, who proposed it in 1919. Here is a formal definition of **Lindahl equilibrium** for the one-public-good, one-private-good case. It's straightforward to write down the definition for multiple public and private goods as well, but that requires more notation than I want to introduce here. We assume here that there is one public good (quantity denoted by x) and one "regular" or private good (with y_i denoting the quantity consumed by i). There are n consumers, with utility functions $u_i(x, y_i)$. There are m firms; each firm has a production function f_j according to which z_j units of input (the private good) are converted into $q_j = f_j(z_j)$ units of the public good. Each consumer i owns the share $\theta_{ij} \geqq 0$ of firm j's profit, and $\Sigma_i \theta_{ij} = 1$ for each $j = 1, \ldots, m$. Denote the price of the private good by p_y. There are **Lindahl prices** (also called **Lindahl taxes**) $p_1, \ldots, p_n$ that the consumers $i = 1, \ldots, n$ are charged for the public good.

Definition: For an economy as described above, a **Lindahl equilibrium** is

a price-list $(p_y^*, p_1^*, \ldots, p_n^*)$,

a production allocation $(z_1^*, \ldots, z_m^*)$, and

a consumption allocation $(x^*, y_1^*, \ldots, y_n^*)$

that satisfy the following conditions, where $p_x^* = \Sigma_1^n p_i^*$:

(π-Max) $\quad \forall j : z_j^*$ maximizes firm j's profit, $\pi_j(z_j) := p_x^* f_j(z_j) - p_y^* z_j$,

(U-Max) $\quad \forall i : (x^*, y_i^*)$ maximizes $u_i(x, y_i)$ subject to $p_i^* x + y_i \leqq \mathring{y}_i + \Sigma_{j=1}^m \theta_{ij} \pi_j(z_j^*)$,

(M-Clr-x) $\quad x^* \leqq \Sigma_1^m q_j^* \quad$ and $\quad x^* = \Sigma_1^m q_j^*$ if $p_x^* > 0$, where $q_j^* = f_j(z_j^*)$, $j = 1, \ldots, m$,

(M-Clr-y) $\quad \Sigma_1^n y_i^* + \Sigma_1^m z_j^* \leqq \Sigma_1^n \mathring{y}_i \quad$ and $\quad \Sigma_1^n y_i^* + \Sigma_1^m z_j^* = \Sigma_1^n \mathring{y}_i$ if $p_y^* > 0$.

Note that this has a certain parallel with the no-externalities Walrasian equilibrium: at both the Walrasian and Lindahl equilibria, the price that a consumer pays for a good is the same for every unit she consumes, and (if she is maximizing utility) the price is equal to her marginal rate of substitution, *i.e.*, her marginal value for the good. But in the Walrasian case everyone pays the *same* price, p, while here everyone will typically be paying a *different* price p_i. The Walrasian equilibrium definition implicitly assumes that the price will adjust if net demands don't sum to zero; the Lindahl equilibrium definition implicitly assumes that the price-shares and quantity will adjust if demands for the public good aren't all the same.

Samuelson argued, colorfully but informally, that the Lindahl idea is unworkable because it's unrealistic to expect people to take their Lindahl prices as given. (See, for example, "The Pure Theory of Public Expenditure", *Review of Economics and Statistics*, 1954.) Arrow, as we will see, provided a clearer, more formal version of this argument, but he drew a less sweeping conclusion from it than Samuelson had done.

The Lindahl equilibrium is useful because it provides a benchmark in which, just as in the Walrasian equilibrium, each consumer's per-unit payment to finance the public good is equal to his marginal value for the good, and no consumer is worse off at the equilibrium than if he instead just consumed his initial endowment, and the resulting allocation is Pareto optimal. These properties have motivated the design of game forms ("institutions") in which the Nash equilibrium actually yields Lindahl prices and a Lindahl allocation.

Arrow's Walrasian Model of Public Goods and Other Externalities

Arrow showed that we can recast the public-goods allocation problem as one involving only private goods, so that our Walrasian analysis applies. Arrow defined each individual's consumption of the public good as a distinct commodity, with a distinct market and price, but with "jointness" in the production of these goods. Here's how this works in our one-public-good-one-private-good model with n consumers (where X is the public good and Y is the private good):

We redefine the economy as having $n+1$ goods $X_1, \ldots, X_n, Y$, with quantities denoted by $x_1, \ldots, x_n, y$. An allocation is therefore an $n(n+1)$-tuple

$$\big((x_1^1, \ldots, x_n^1, y^1), (x_1^2, \ldots, x_n^2, y^2), \ldots, (x_1^n, \ldots, x_n^n, y^n)\big) \in \mathbb{R}_+^{n(n+1)}.$$

However, both the production possibilities and the consumption possibilities in this economy are assumed to have a special character:

(1) The X-goods are "joint products" in any firm's production process: A production plan for a firm is an $(n+1)$-tuple $(z, \mathbf{q}) = (z, q_1, \ldots, q_n) \in \mathbb{R}_+^{n+1}$, where z is the amount of the private good the firm uses as input and q_i is the output of commodity X_i, but the firm has the technological constraint $q_1 = q_2 = \cdots = q_n$. This is exactly like the classical joint products mutton and wool that are produced by raising sheep.

(2) Consumer i's consumption set is $\{(x_1^i, \ldots, x_n^i, y^i) \in \mathbb{R}_+^{n+1} \mid j \neq i \Rightarrow x_j^i = 0\}$ — *i.e.*, Consumer i can consume only the goods X_i and Y. So while Consumer i's utility function u^i is technically defined on the domain $\mathbb{R}_+^{n+1}$, we can more intuitively write u^i as defined on bundles $(x^i, y^i) \in \mathbb{R}_+^2$. Therefore we can simplify the notation, defining an allocation to consumers as a $2n$-tuple $(x_i, y_i)_1^n \in \mathbb{R}_+^{2n}$.

Now a Lindahl equilibrium is just a Walrasian equilibrium of this joint-product economy. Specifically (and assuming for simplicity that there is just a single producer/firm, which is a price-taker), a Walrasian equilibrium is a price-list $(\widehat{p}_1, \ldots, \widehat{p}_n, \widehat{p}_y) \in \mathbb{R}_+^{n+1}$, a consumption allocation $(\widehat{x}_i, \widehat{y}_i)_1^n \in \mathbb{R}_+^{2n}$ and a production plan $(\widehat{z}, \widehat{q}_1, \ldots, \widehat{q}_n) \in \mathbb{R}_+^{n+1}$ that satisfy

(U-max) $\quad \forall i : (\widehat{x}_i, \widehat{y}_i)$ maximizes $u^i(x_i, y_i)$ subject to $\widehat{p}_i x_i + y_i \leqq \mathring{y}_i + \theta_i \pi(\widehat{z}, \widehat{\mathbf{q}})$

(π-max) $\quad (\widehat{z}, \widehat{\mathbf{q}})$ maximizes $\pi(z, q_1, \ldots, q_n) = \sum_{i=1}^n \widehat{p}_i q_i - \widehat{p}_y z$ subject to $q_1 = \cdots q_n = f(z)$

(M-Clr) $\quad \forall i : \widehat{x}_i = \widehat{q}_i \quad$ and $\quad \widehat{z} + \sum_{i=1}^n \widehat{y}_i \leqq \sum_{i=1}^n \mathring{y}_i$, with equality if $\widehat{p}_y > 0$.

Therefore the First Welfare Theorem applies: if the utility functions and production functions satisfy the usual assumptions, then the equilibrium allocation will be Pareto efficient.

But Arrow's model also makes it clear that the Walrasian models's price-taking assumption for consumers is unrealistic here: for each of the distinct goods X_i there is only one person on the demand side of the market. The only person who cares about the good X_i is person i. It's clearly unrealistic to assume that any of the participants will take their own price (or Lindahl cost share) as given. This was Arrow's motivation for modeling things this way — to clarify this point.

Game Forms and Mechanism Design

Recall that a **game** is an n-tuple $(S_i, \pi_i)_{i=1}^n$, where

S_i is i's strategy or action set $(i = 1, \ldots, n)$,

$\pi_i : S_1 \times \cdots \times S_n \to \mathbb{R}$ is i's payoff function $(i = 1, \ldots, n)$.

A **game form** is a way to model the rules of a game, or an institution, independently of the players' utility functions over the game's outcomes. The notion of a game form is an important idea for *mechanism design* (also called *institution design* or *market design*).

Definition: Let X be a set of possible **outcomes**. A **game form** for X consists of

(1) n **action sets** $S_1, \ldots, S_n$, and

(2) an **outcome function** $\varphi : S_1 \times \cdots \times S_n \to X$.

Definition: Given an outcomes set X and

(1) a game form $(S_1, \ldots, S_n; \varphi)$ for X, and

(2) n utility functions $u_i : X \to \mathbb{R}$ over outcomes $(i = 1, \ldots, n)$,

the **associated game** or **induced game** is defined by the n action sets $S_1, \ldots, S_n$ and the n payoff functions

$$\widetilde{u}_i(s_1, \ldots, s_n) := u_i(\varphi(s_1, \ldots, s_n)),\ i = 1, \ldots, n.$$

In our public goods model, where x is the level at which the public good is provided and y_i is the number of dollars i spends on other goods, an outcome is an $(n+1)$-tuple $(x, y_1, \ldots, y_n) \in \mathbb{R}_+^{n+1}$, so our outcome set is $X = \mathbb{R}_+^{n+1}$. Assume that the cost of the public good is given by $C(x) = cx$, so marginal cost is c (for example, c is the price that's charged for each unit of the public good).

Example: The Voluntary Contributions Mechanism (VCM) for a public good.

The **VCM** institution, or game form, is defined by the following action sets and outcome function:

Actions: Each person i chooses a **contribution** m_i in the action set $\mathbb{R}_+$. Let $\mathbf{m} = (m_1, \ldots, m_n)$.

Outcome function:

$x = \pi(\mathbf{m}) = \frac{1}{c}\sum_1^n m_i$ (*i.e.*, x is whatever quantity the contributions $\sum_1^n m_i$ will buy);

$y_i = \mathring{y}_i - t_i$, where $t_i = \tau^i(\mathbf{m}) = m_i$ (*i.e.*, i's "tax" is simply his contribution, m_i) .

Thus, the outcome function is $\varphi(\mathbf{m}) = (\pi(\mathbf{m}), \mathring{y}_1 - \tau^1(\mathbf{m}), \ldots, \mathring{y}_n - \tau^n(\mathbf{m}))$.

The induced game is given by the utility functions $u^i(x, y_i),\ i = 1, \ldots, n$, so the payoff functions in the induced game are

$$\widetilde{u}^i(m_1, \ldots, m_n) := u^i(\pi(\mathbf{m}),\ \mathring{y}_i - \tau^i(\mathbf{m})) = u^i\Big(\frac{1}{c}\sum_{j=1}^n m_j,\ \mathring{y}_i - m_i\Big),\ i = 1, \ldots, n.$$

The Nash equilibrium of the VCM institution (*i.e.*, the NE of the associated game) is as follows:

The first-order marginal condition that characterizes individual i's choice of m_i is

$$\text{(FOMC)} \qquad \frac{\partial \widetilde{u}^i}{\partial m_i} \leqq 0 \quad \text{and} \quad \frac{\partial \widetilde{u}^i}{\partial m_i} = 0 \text{ if } m_i > 0.$$

We have

$$\frac{\partial \widetilde{u}^i}{\partial m_i} = \frac{\partial u^i}{\partial x_i}\frac{\partial \pi}{\partial m_i} + \frac{\partial u^i}{\partial y_i}\frac{\partial(-\tau^i)}{\partial m_i} = u_x^i \cdot \frac{1}{c} + u_y^i \cdot (-1) = \frac{1}{c}u_x^i - u_y^i.$$

Therefore

$$\frac{\partial \widetilde{u}^i}{\partial m_i} \leqq 0 \quad \text{if and only if} \quad \frac{u_x^i}{u_y^i} \leqq c.$$

Therefore the FOMC above, for individual i, can be written as

$$\frac{u_x^i}{u_y^i} \leqq c \quad \text{and} \quad \frac{u_x^i}{u_y^i} = c \text{ if } m_i > 0$$

$$\textit{i.e.,} \qquad MRS^i \leqq MC \quad \text{and} \quad MRS^i = MC \text{ if } m_i > 0.$$

Note that this is identical to the *market outcome* we obtained earlier, in which the public good is provided at a level that's less than the Pareto level: those who contribute are only those with the largest MRS^i; everyone else is a free rider; and *no one* will contribute if everyone has $MRS^i < MC$ when $x = 0$.

Mechanism Design: The mechanism design problem is to devise an outcome function φ for which the Nash equilibria (or some other specified solution) have one or more desirable properties — for example, an outcome function for which the Nash equilibria are Pareto efficient. For our simple public-goods model, the outcome function φ is the $(n+1)$-tuple of functions $(\pi, \tau^1, \ldots, \tau^n)$, so our mechanism design problem is to devise a provision function π and tax/transfer functions τ^i for each i for which the Nash equilibrium is Pareto efficient, or better yet, is a Lindahl equilibrium allocation.

The first institution/mechanism with Pareto efficient Nash equilibria was devised by Grove & Ledyard. The first mechanism with Lindahl Nash equilibria was devised by Leo Hurwicz.

DESIGNING A MECHANISM WITH PARETO EFFICIENT NASH EQUILIBRIA

(A MECHANISM TO CHOOSE THE PROVISION LEVEL AND FINANCING OF A PUBLIC GOOD)

THE MECHANISM HAS TO CHOOSE $(x; t_1, \dots, t_n) \in \mathbb{R}_+ \times \mathbb{R}^n$.
LET'S ALLOW PARTICIPANTS TO CHOOSE $m_i \in \mathbb{R}$ $(i = 1, \dots, n)$.
A CONFIGURATION OF ALL n CHOICES IS AN $m \in \mathbb{R}^n$.
LET'S DENOTE THE OUTCOME FUNCTIONS, OR "RULES," AS:

$x = \pi(m)$, THE "PROVISION FUNCTION"

$y_i = \mathring{y}_i - t_i$, WHERE $t_i = \tau^i(m)$, i's "TAX RULE."

IF WE WANT THE NE TO BE PARETO EFFICIENT, WHAT CONDITIONS WILL THE FUNCTIONS $\pi, \tau^1, \dots, \tau^n$ HAVE TO SATISFY?

WRITE i's PAYOFF FUNCTION AS $\tilde{u}^i(m) := u^i(\pi(m), \mathring{y}_i - \tau^i(m))$.

AT AN INTERIOR NE WE MUST HAVE $\frac{\partial \tilde{u}^i}{\partial m_i} = 0$. BUT

$$\frac{\partial \tilde{u}^i}{\partial m_i} = u^i_x \pi_i - u^i_y \tau^i_i, \text{ WHERE } \underbrace{u^i_x = \partial u^i/\partial x,\ u^i_y = \partial u^i/\partial y,\ \pi_i = \partial \pi/\partial m_i,\ \tau^i_i = \partial \tau^i/\partial m_i.}$$

$$\therefore \frac{\partial \tilde{u}^i}{\partial m_i} = 0 \iff \frac{u^i_x}{u^i_y} = \frac{\tau^i_i}{\pi_i}, \text{ i.e., } \boxed{MRS^i = \frac{\tau^i_i}{\pi_i}}$$

SINCE EFFICIENCY REQUIRES $\sum MRS^i = MC$ (IN THE INTERIOR), WE WILL HAVE TO HAVE

$$\boxed{\sum_{i=1}^n \frac{\tau^i_i}{\pi_i} = MC.}$$

IF THE OTHERS' ACTIONS (i.e., THE $(n-1)$-TUPLE $m_{\sim i}$) ARE TAKEN AS GIVEN, THEN THE CHOICE OF m_i BY INDIVIDUAL i IMPLIES A CHOICE OF BOTH x AND y_i. OF COURSE, HIS CHOICE IS CONSTRAINED TO SATISFY SOME KIND OF TRADE-OFF BETWEEN x AND y_i — i.e., THE SET OF (x, y_i) PAIRS AVAILABLE BY VARYING THE CHOICE m_i IS CONSTRAINED, AND THE TRADE-OFF IS THE SLOPE OF THAT CONSTRAINT.

THE PROTOTYPE FOR THIS IDEA IS THE FAMILIAR MARKET-BASED BUDGET CONSTRAINT, DEFINED BY THE FACT THAT EXPENDITURE $E(x,y)$ ON x AND y CANNOT EXCEED (AND THUS IS SET EQUAL TO) INCOME OR WEALTH, M:

$$E(x,y) = M$$

$\therefore$ IF x AND y ARE ADJUSTED, BUT SO AS TO REMAIN ON THIS CONSTRAINT, WE HAVE

$$\Delta E = 0; \quad \text{i.e.,} \quad \frac{\partial E}{\partial x}\Delta x + \frac{\partial E}{\partial y}\Delta y \cong 0;$$

$$\text{i.e.,} \quad \frac{\Delta y}{\Delta x} \cong -\frac{\frac{\partial E}{\partial x}}{\frac{\partial E}{\partial y}}; \quad \text{i.e.,} \quad -\frac{\partial y}{\partial x} = \frac{E_x}{E_y}.$$

IF $E(x,y) := p_x x + p_y y$,

THEN THIS BECOMES

$$p_x \Delta x + p_y \Delta y \cong 0 \quad \text{AND} \quad -\frac{\partial y}{\partial x} = \frac{p_x}{p_y}.$$

FOR A MECHANISM DEFINED BY $\pi(m)$ AND $\tau^i(m)$:

$$\Delta y_i \cong -\tau^i_i \Delta m_i \quad \text{AND} \quad \Delta x \cong \pi_i \Delta m_i$$

$$\text{WHERE } \tau^i_i := \frac{\partial \tau^i}{\partial m_i} \quad \text{AND} \quad \pi_i := \frac{\partial \pi}{\partial m_i}.$$

$$\therefore \frac{\Delta y_i}{\Delta x} \cong -\frac{\tau^i_i \Delta m_i}{\pi_i \Delta m_i} = -\frac{\tau^i_i}{\pi_i}$$

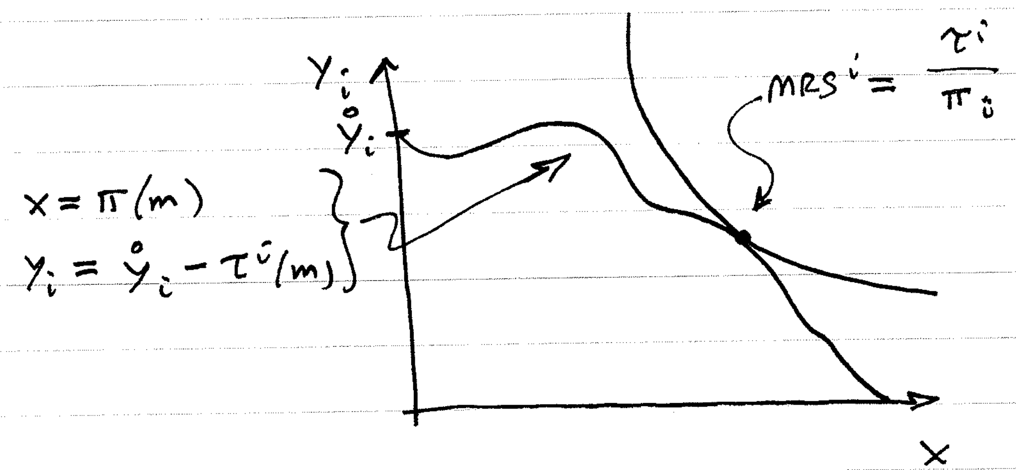
$$\text{i.e., } -\frac{\partial y_i}{\partial x} = \frac{\tau^i_i}{\pi_i}$$

... THE SLOPE OF THE CONSTRAINT ON THE BUNDLE (x, y_i) IS $-\dfrac{\tau^i_i}{\pi_i}$.

EQUIV.: WRITE $y_i = \mathring{y}_i - \tau^i(\pi^{-1}(x))$

$\therefore \dfrac{\partial y_i}{\partial x} = -\tau^i_i \dfrac{1}{\pi_i}$

[π^{-1} IS THE m_i THAT YIELDS x, GIVEN m_{-i}]



INDIVIDUAL i'S CHOICE OF m_i WILL THEREFORE SATISFY THE MARGINAL CONDITION

$$\boxed{MRS^i = \frac{\tau^i_i}{\pi_i}.}$$

EXAMPLE 1: GROVES-LEDYARD "QUADRATIC" MECHANISM

$$x = \pi(m) \quad \text{AND} \quad y_i = \overset{\circ}{y}_i - \tau^i(m).$$

$$\pi(m) = \bar{m} = \frac{1}{n}\sum_{j=1}^{n} m_j$$

$$\tau^i(m) = \frac{1}{n}C(\pi(m)) + P^i(m) - R(m),$$ (ξ IS GIVEN)

WHERE $P^i(m) := \xi(m_i - \bar{m})^2$, A "PENALTY"

$R(m) := \frac{1}{n}\sum_1^n P^j(m)$, A "REBATE."

WE HAVE $\pi_i = \frac{1}{n}$ AND

$$\tau^i_i = \frac{1}{n^2}mc + P^i_i - \frac{1}{n}\sum_1^n P^j_i,$$

WHERE $P^i_i = 2\xi(m_i - \bar{m})\left(1 - \frac{1}{n}\right)$

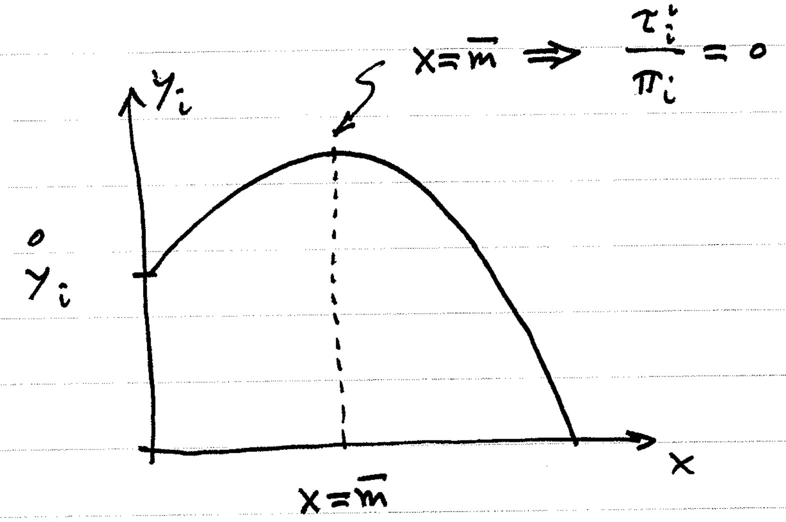
$P^j_i = 2\xi(m_j - \bar{m})\left(-\frac{1}{n}\right), \quad j \neq i.$

$$\therefore \tau^i_i = \frac{1}{n^2}mc + \frac{n-1}{n}2\xi(m_i - \bar{m}) +$$
$$+ \frac{1}{n^2}2\xi\underbrace{\sum_{j=1}^{n}(m_j - \bar{m})}_{\equiv 0} - \frac{1}{n}2\xi(m_i - \bar{m})$$

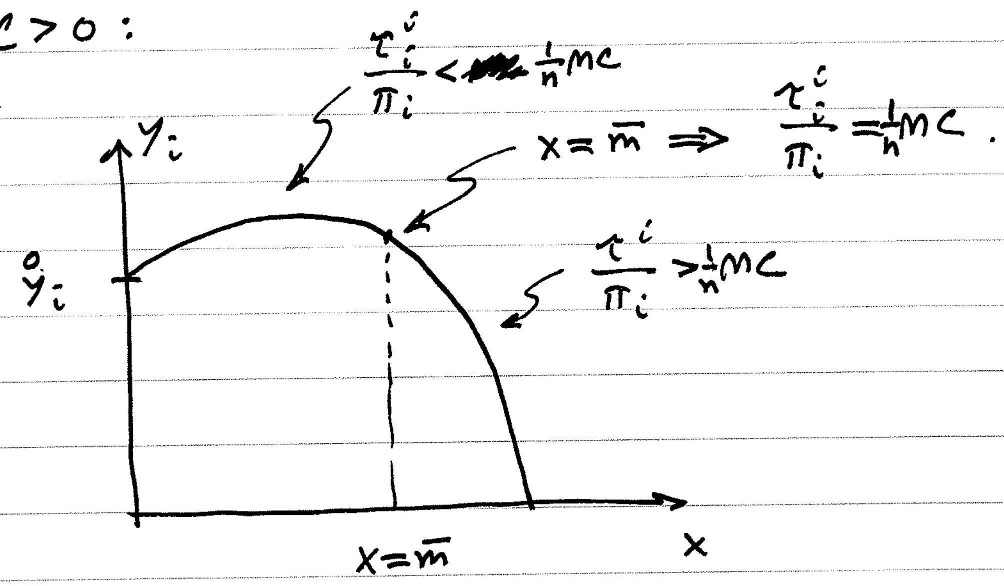
$$= \frac{1}{n^2}mc + 2\xi(m_i - \bar{m})\left(\frac{n-2}{n}\right).$$

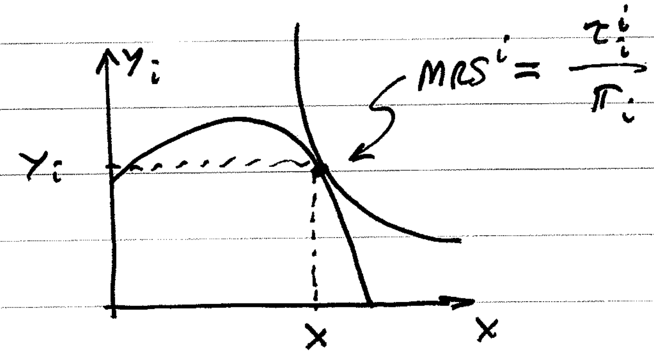
$$\therefore \frac{\tau^i_i}{\pi_i} = \frac{1}{n}mc + 2(n-2)\xi(m_i - \bar{m});$$ THE SLOPE OF THE CONSTRAINT INCREASES LINEARLY AS WE MOVE AWAY FROM $x = \bar{m}$; $\therefore$ THE CONSTRAINT IS QUADRATIC. $\left[\text{NOTE: } \sum \frac{\tau^i_i}{\pi_i} \equiv mc\right]$

THE CASE $MC = 0$:



IF $MC > 0$:





EXAMPLE 2: WALKER

ASSUME $C(x) = \beta x$. (JUST TO SIMPLIFY)

$\pi(m) = \sum m_j$

$\tau^i(m) = p_i(m)\, x = p_i(m)\, \pi(m)$,

WHERE $p_i(m) = \frac{1}{n}\beta + \xi(m_{i+2} - m_{i+1})$; $\begin{cases} n+1 := 1 \\ n+2 := 2 \end{cases}$.

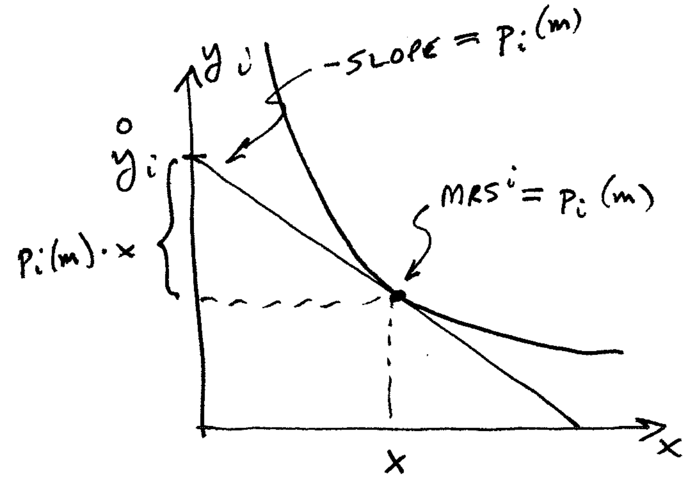
NOTE THAT $p_i(m)$ DOES NOT DEPEND ON m_i:
IF i TAKES $m_{\sim i}$ AS GIVEN, THEN HE TAKES $p_i(m)$ AS GIVEN.

WE HAVE $\tau^i_i := \frac{\partial \tau^i}{\partial m_i} = p_i(m) + \frac{\partial p_i}{\partial m_i}\sum m_j = p_i(m)$. !

$\therefore\ \frac{\tau^i_i}{\pi_i} = p_i(m)$; $\therefore$ i CHOOSES m_i S.T. $MRS^i = p_i(m)$.

$\therefore\ \sum MRS^i = \sum p_i(m) = \beta + \xi \sum_1^n (m_{i+1} - m_{i+2}) \equiv \beta = MC$. !

$\therefore$ THE OUTCOME IS PO. BUT ALSO $t_i = (MRS^i)\cdot x$,
SO THIS A LINDAHL EQUILIBRIUM OUTCOME.



BUT WE HAVE REASON TO BE SKEPTICAL OF THE LINDAHL OUTCOME.

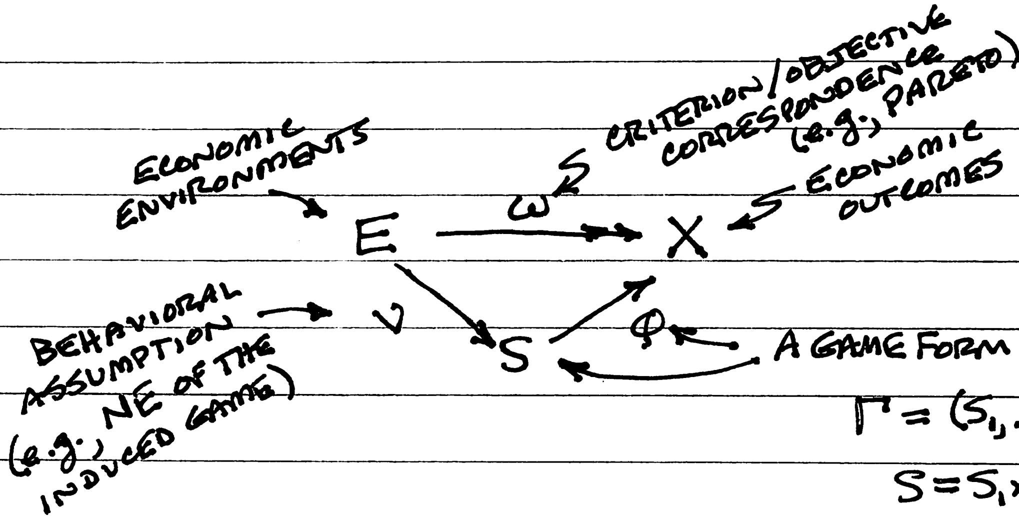
THERE IS SOME REASON TO BELIEVE THAT ...

(a) THIS MECHANISM IS UNSTABLE ;

(b) PARTICIPANTS WON'T TAKE $m_{\sim i}$ AS "GIVEN."

THIS CALLS INTO QUESTION THE SIMPLE RELIANCE ON NE.

MECHANISM DESIGN: THE REITER DIAGRAM



$\Gamma = (S_1, \ldots, S_n; \varphi)$

$S = S_1 \times \cdots \times S_n$

$\varphi : S \to X$

THE SETS E AND X AND THE CORRESPONDENCE ω ARE GIVEN, AND EACH $e \in E$ INCLUDES (AT LEAST) UTILITY FUNCTIONS $u^i : X \to \mathbb{R}$ $(i = 1, \ldots, n)$.

THE FUNCTION (OR CORRESPONDENCE) ν DESCRIBES OUR ASSUMPTION ABOUT WHAT CHOICES $s_i \in S_i$ THE PARTICIPANTS WILL MAKE IN EACH $e \in E$.

THE GAME FORM $\Gamma = (S_1, \ldots, S_n; \varphi)$ IS THE "VARIABLE" IN THE MECHANISM DESIGN PROBLEM, WHICH CAN BE CHOSEN. WE WISH TO CHOOSE A Γ THAT SATISFIES $\varphi(\nu(e)) = \omega(e)$ OR $\varphi(\nu(e)) \in \omega(e)$ OR $\varphi(\nu(e)) \subseteq \omega(e)$ FOR EACH $e \in E$.

IN OUR PUBLIC GOOD PROBLEM:

$x \in X$: $x = (x, y_1, \ldots, y_n) \in \mathbb{R}^{n+1}_+$.

$e \in E$: $e = (u^i)_1^n$ OR $e = (u^i; \mathring{y}_i)_1^n$.

$s \in S$: WE DECIDED TO SAY $s_i = m_i \in \mathbb{R}$ OR $\mathbb{R}_+$; $S_i = \mathbb{R}$ OR $\mathbb{R}_+$

φ: WE DERIVED CONDITIONS φ WOULD HAVE TO SATISFY TO ENSURE $\forall e \in E$: $\varphi(\nu(e))$ IS PARETO FOR e.

Externalities and the "Coase Theorem"

The "Coase Theorem" has been one of the most influential contributions to come from economics in the last fifty years. Its influence on the law has been especially profound.

The so-called "theorem" goes something like this:

"If property rights and liabilities for an activity are fully assigned, then an efficient outcome will result, even in the presence of externalities. Moreover, the level at which the activity is carried out will not depend on the particular assignment of rights and liabilities."

An Externality Example:

Suppose one party engages in an activity that produces a **benefit** for itself, but which **damages** other parties. Let's suppose that the first party is a firm, whose production activity generates profit but also pollutes the air or water, adversely affecting one or more nearby residents, or negatively affecting the profitability of one or more nearby firms. In the simplest case, let's suppose that it's just a single consumer who suffers damages from the firm's production. Let x denote the level of the externality-generating activity, in this case the firm's production; let $B(x)$ denote the resulting benefit to the firm (its profit); and let $D(x)$ denote the damage to the consumer (for example, the amount he would be willing to pay to eliminate the effects the externality imposes on him). We're assuming here that both the benefit and the damage are measured in dollars, and we'll assume that the net social benefit of operating the activity at level x is $v(x) = B(x) - D(x)$. The socially (Pareto) optimal level of the activity is therefore the level x at which $B'(x) = D'(x)$, *i.e.*, where the marginal benefit equals the marginal damage. Coase emphasized that the optimal level of the damaging activity is *not* zero, and he described how the two parties should be expected to bargain to an outcome in which they agree that the activity will be operated at the optimal level, with one party paying compensation to the other — just as we described in our public-good example of the water level in the lake.

In fact, there is no formal difference between this situation and our earlier two-person public good problem. Figure 1, below, depicts the "Kolm diagram" for this situation. Note that in the core allocations the externality-generating activity is always operated at the Pareto level, but that both the direction and the size of the compensation depend upon which party has the right to set the level of x. In this example, then, the Coase Theorem could easily be formalised as an actual theorem.

But the simplicity of the example also allows us to see a number of caveats, any of which would change the Coase result, in some cases by only a little, but in others it would be significantly changed:

(1) If the objective of either party has income effects, then the level of the activity will not be determinate, and in particular it will depend upon which party has the right to set the level of x and would therefore receive compensation. This effect would not typically be large.

(2) If there is a large number of individuals on either or both sides of the "market," then the bargaining or transaction costs can become prohibitively large, making it unlikely that a core (or Pareto) outcome will be achieved by bargaining.

(3) If there is a large number of individuals on either side of the market, then the core, unlike in the no-externalities case, is not small. The various core allocations will differ considerably in the distribution of the compensations paid and/or received, further increasing the difficulties in even achieving a core allocation.

Why has the "Coase Theorem" been so influential? Most likely it's because Coase's article ("The Problem of Social Cost") was published in the *Journal of Law & Economics*, and was written in the style of law journals — no mathematical symbols or equations; every idea presented as an actual externality that had arisen in legislative or court cases (albeit always a two-party externality); and descriptions and citations of actual legal cases and case law on every page. It's had an enormous impact on the legal profession. In fact, by some measures it's the most-cited law review article of all time, with from 40% to 80% more citations than the second-most-cited.

The effect of Coase's paper has been that when microeconomics predicts that an outcome will be inefficient (or not in the core), we're likely to ask ourselves whether the participants would themselves devise some means to overcome the inefficiency. In both the legal and economics professions, the focus now tends to be on the barriers to efficient outcomes, and why the barriers can't be overcome. This focus on barriers to efficiency, and on whether the barriers are likely to be overcome without intervention, is not restricted to externalities and public goods. Other examples are incomplete markets for dealing with uncertainty; adverse selection; and moral hazard.

Incidentally, Coase himself did not describe this idea as a theorem, and he was apparently not happy that is was described that way. His paper pointed out that the felicitous outcome he was describing would often not be achieved, due to the attendant transaction costs.

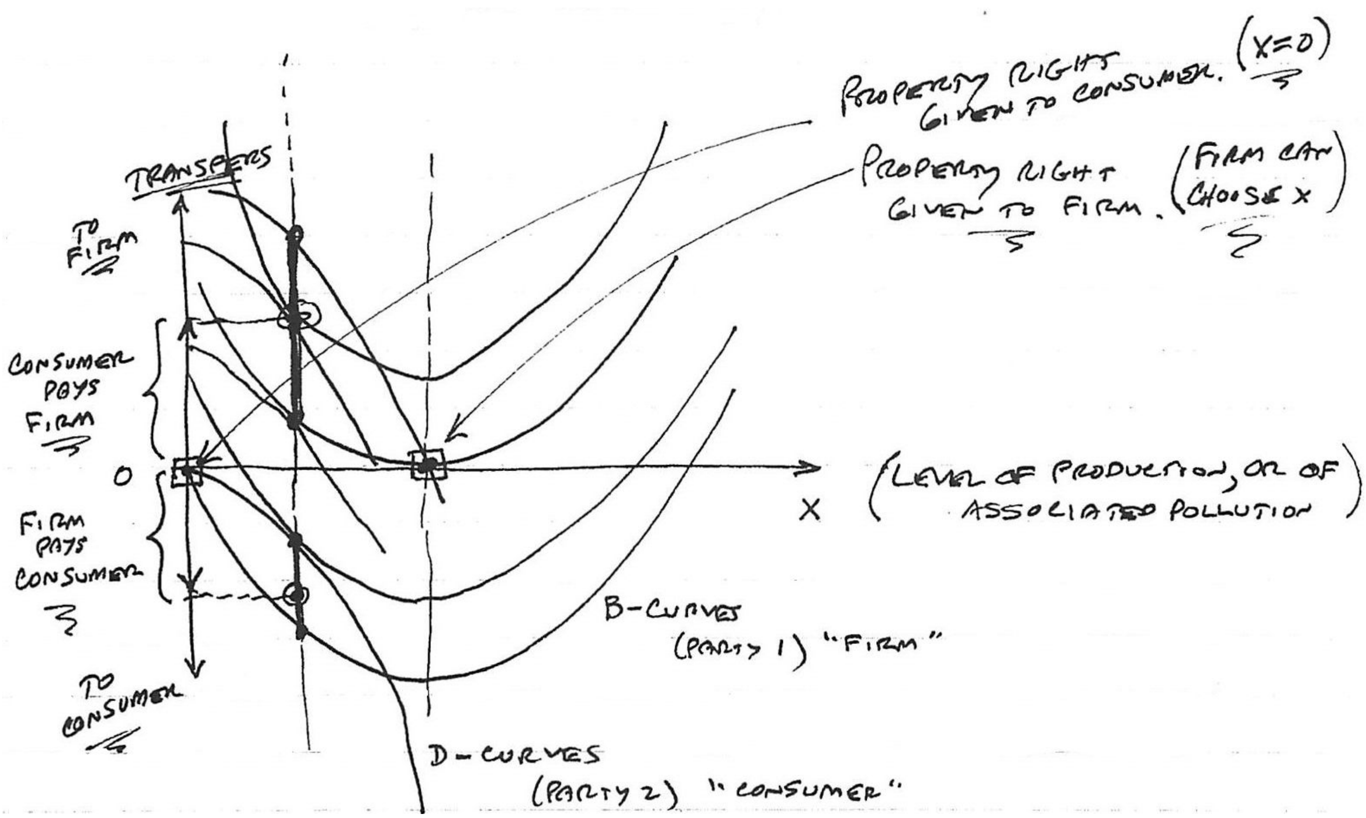

PROPERTY RIGHT GIVEN TO CONSUMER. (X=0)
PROPERTY RIGHT GIVEN TO FIRM. (FIRM CAN CHOOSE X)
TRANSFERS
TO FIRM
CONSUMER PAYS FIRM
0
FIRM PAYS CONSUMER
TO CONSUMER
X
(LEVEL OF PRODUCTION, OR OF ASSOCIATED POLLUTION)
B-CURVES (PARTY 1) "FIRM"
D-CURVES (PARTY 2) "CONSUMER"

Figure 1

Externalities: Pigovian Taxes and Subsidies

We'll model a situation in which the production of a consumption good X generates external effects on consumers. Examples are air pollution, water pollution, noise, etc. Let x denote quantities of the X good and let y denote dollars or quantities of a good Y that's a composite of all other goods. Let s denote the level of the externality.

Consumers:

There are n consumers, each represented by a utility function u^i and an endowment bundle $(\mathring{x}_i, \mathring{y}_i)$. We assume that $\mathring{x}_i = 0$ and $\mathring{y}_i > 0$. Each consumer's utility function has the form $u^i(x_i, y_i, s)$, where s is the level of the externality. We'll express marginal rates of substitution in terms of the Y good. Note that the externality is "good" for consumer i if $MRS_s^i > 0$ and is "bad" for i if $MRS_s^i < 0$.

Production:

There are m firms that can produce X, each one according to a production function $q_j = f_j(z_j)$, where z_j is the amount of the Y good firm j uses as input. Therefore we have $\sum_{j=1}^m z_j = \sum_{i=1}^n (\mathring{y}_i - y_i)$. We assume that $s = q = \sum_{j=1}^m q_j$; results are the same if $s = g(q)$.

We'll assume that all utility functions are increasing in x and y, that all production functions are increasing, and that all utility functions and production functions are continuously differentiable and concave.

Pareto Efficiency:

The Pareto maximization problem is

$$\max \sum_{i=1}^n \lambda_i u^i(x_i, y_i, s) \text{ subject to } x_i, y_i, z_j \geqq 0 \ \ \forall i, j \text{ and}$$

$$\sum_i x_i \leqq \sum_j f_j(z_j) \qquad (\sigma_x)$$

$$\sum_i y_i + \sum_j z_j \leqq \sum_i \mathring{y}_i \qquad (\sigma_y)$$

$$\sum_j f_j(z_j) \leqq s \qquad (\sigma_s)$$

The first-order marginal conditions at an interior solution are

$$\begin{array}{lrcll} x_i: & \lambda_i u_x^i & = & \sigma_x \quad \forall i & (1) \\ y_i: & \lambda_i u_y^i & = & \sigma_y \quad \forall i & (2) \\ s: & \sum_i \lambda_i u_s^i & = & -\sigma_s & (3) \\ z_j: & 0 & = & \sigma_y + \sigma_s f_j'(z_j) - \sigma_x f_j'(z_j) & (4) \end{array}$$

Equations (1) and (2) yield

$$\forall i: \ \frac{u_x^i}{u_y^i} = \frac{\sigma_x}{\sigma_y}, \quad i.e., \ MRS_x^i = \frac{\sigma_x}{\sigma_y}, \tag{5}$$

and equations (2) and (3) yield

$$\sigma_y \sum_{i=1}^{n} \frac{u_s^i}{u_y^i} = -\sigma_s, \quad i.e., \quad \sum_{i=1}^{n} MRS_s^i = \frac{\sigma_s}{\sigma_y}. \tag{6}$$

Equations (4) can be rewritten as

$$\forall j: \ (\sigma_x - \sigma_s) f_j'(z_j) = \sigma_y, \quad i.e., \ \frac{\sigma_x}{\sigma_y} - \frac{\sigma_s}{\sigma_y} = \frac{1}{f_j'(z_j)}. \tag{7}$$

Combining equations (5), (6), and (7), we have

$$\forall i, j: \ MRS_x^i = MC^j - \sum_{h=1}^{n} MRS_s^h. \tag{$*$}$$

Recall that $MRS_s^i < 0$ if the externality is bad for consumer i and $MRS_s^i > 0$ if the externality is good for consumer i. Therefore the sum $\sum_1^n MRS_s^i$ represents the net marginal benefit of the externality, aggregated over all consumers. If $\sum_1^n MRS_s^i < 0$, then the marginal conditions $(*)$ tell us that Pareto efficiency requires each consumer's marginal value for the x-good to be equal to the good's marginal social cost — the marginal cost of producing another unit of it, plus $|\sum_1^n MRS_s^i|$, the aggregate marginal damage the consumers suffer from producing another unit of the x-good. On the other hand, if the net effect of the externality is positive — *i.e.*, $\sum_1^n MRS_s^i > 0$ — then Pareto efficiency requires that each consumer's marginal value for the good be less than its marginal production cost by the net amount of the marginal indirect benefit consumers receive via the externality, $\sum_1^n MRS_s^i$.

The Pigovian Tax:

Now suppose the net marginal externality associated with producing the x-good is negative — *i.e.*, $\sum_1^n MRS_s^i < 0$ — and suppose that in the market for the x-good all the consumers and producers are price-takers. At an equilibrium, then, we would have $MRS^i = p_x = MC^j$ for each consumer i and each firm j. Therefore too much of the x-good is being produced (recall that each u^i is increasing in x and y, so each MRS_x^i is decreasing in x): everyone's marginal value for the good is less than its social cost of production $MC^j - \sum_{i=1}^n MRS_s^i = MC^j + |\sum_{i=1}^n MRS_s^i|$, so Pareto efficiency requires that production be reduced.

As a solution to this inefficiency, the early-20th-century English economist A.C. Pigou developed what we now call a **Pigovian tax**. Let t denote the level of a per-unit tax imposed on purchases of the x-good, and let t be equal to $-\sum_{i=1}^n MRS_s^i$, the net marginal damages generated by the marginal unit of the x-good produced, at the Pareto efficient level of production and consumption. Now we should expect a consumer to purchase the good to the point where her MRS^i is equal to $p_x + t$ — *i.e.*, $MRS^i = MC^j + |\sum_{i=1}^n MRS_s^i|$, thereby satisfying the efficiency condition $(*)$. If the externality is beneficial — $\sum_1^n MRS_s^i > 0$ — we still set $t = -\sum_{i=1}^n MRS_s^i$, so in this case t is a per-unit subsidy paid to purchasers of the x-good.

Note that in either case — a negative or a positive externality — it's straightforward to balance the budget. In the case of a negative externality the aggregate of all the taxes collected, $\sum_{i=1}^n t_i x_i$, can be rebated to consumers as lump-sum per capita payments. In the case of a positive externality, consumers can be charged a lump-sum per capita tax (for example) to finance the per-unit subsidies, which total $\sum_{i=1}^n |t_i x_i|$.

Determining $\sum_1^n \mathbf{MRS_s^i}$:

The Pigovian analysis leaves open the question of how we can determine, or at least estimate, the value of $\sum_1^n MRS_s^i$. An approach called **contingent valuation** has been developed to accomplish this. The contingent valuation procedure first samples the relevant population: the individuals in the sample are asked to report how much (in dollar terms) they would value some particular increase or decrease in the amount of the externality s. This tells us the MRS_s^i for each individual i in the sample. Then the sample is used to calculate an estimate of $\sum_1^n MRS_s^i$, based on the demographics of the sample.

There is an obvious problem with this procedure, however. If the individuals are paid an amount that is based on their reported MRS_s^i, they will have an incentive to report very large negative values of MRS_s^i, in order to receive large payments. On the other hand, if they're *not* paid, each person's incentive is to report an MRS_s^i that will move the level of s in the direction he prefers, given his belief about the values of MRS_s reported by others in the sample.