Arrow's Walrasian Model
of Public Goods and Other Externalities

Arrow showed that we can recast the public-goods allocation problem as one involving only private goods, so that our Walrasian analysis applies. Arrow defined each individual's consumption of the public good as a distinct commodity, with a distinct market and price, but with "jointness" in the production of these goods. Here's how this works in our one-public-good-one-private-good model with n consumers (where X is the public good and Y is the private good):

We redefine the economy as having $n+1$ goods $X_1, \ldots, X_n, Y$, with quantities denoted by $x_1, \ldots, x_n, y$. An allocation is therefore an $n(n+1)$-tuple

$$\big((x_1^1, \ldots, x_n^1, y^1), (x_1^2, \ldots, x_n^2, y^2), \ldots, (x_1^n, \ldots, x_n^n, y^n)\big) \in \mathbb{R}_+^{n(n+1)}.$$

However, both the production possibilities and the consumption possibilities in this economy are assumed to have a special character:

(1) The X-goods are "joint products" in any firm's production process: A production plan for a firm is an $(n+1)$-tuple $(z, \mathbf{q}) = (z, q_1, \ldots, q_n) \in \mathbb{R}_+^{n+1}$, where z is the amount of the private good the firm uses as input and q_i is the output of commodity X_i, but the firm has the technological constraint $q_1 = q_2 = \cdots = q_n$. This is exactly like the classical joint products mutton and wool that are produced by raising sheep.

(2) Consumer i's consumption set is $\{(x_1^i, \ldots, x_n^i, y^i) \in \mathbb{R}_+^{n+1} \mid j \neq i \Rightarrow x_j^i = 0\}$ — *i.e.*, Consumer i can consume only the goods X_i and Y. So while Consumer i's utility function u^i is technically defined on the domain $\mathbb{R}_+^{n+1}$, we can more intuitively write u^i as defined on bundles $(x^i, y^i) \in \mathbb{R}_+^2$. Therefore we can simplify the notation, defining an allocation to consumers as a $2n$-tuple $(x_i, y_i)_1^n \in \mathbb{R}_+^{2n}$.

Now a Lindahl equilibrium is just a Walrasian equilibrium of this joint-product economy. Specifically (and assuming for simplicity that there is just a single producer/firm, which is a price-taker), a Walrasian equilibrium is a price-list $(\widehat{p}_1, \ldots, \widehat{p}_n, \widehat{p}_y) \in \mathbb{R}_+^{n+1}$, a consumption allocation $(\widehat{x}_i, \widehat{y}_i)_1^n \in \mathbb{R}_+^{2n}$ and a production plan $(\widehat{z}, \widehat{q}_1, \ldots, \widehat{q}_n) \in \mathbb{R}_+^{n+1}$ that satisfy

(U-max) $\quad \forall i : (\widehat{x}_i, \widehat{y}_i)$ maximizes $u^i(x_i, y_i)$ subject to $\widehat{p}_i x_i + y_i \leqq \mathring{y}_i + \theta_i \pi(\widehat{z}, \widehat{\mathbf{q}})$

(π-max) $\quad (\widehat{z}, \widehat{\mathbf{q}})$ maximizes $\pi(z, q_1, \ldots, q_n) = \sum_{i=1}^n \widehat{p}_i q_i - \widehat{p}_y z$ subject to $q_1 = \cdots q_n = f(z)$

(M-Clr) $\quad \forall i : \widehat{x}_i = \widehat{q}_i \quad$ and $\quad \widehat{z} + \sum_{i=1}^n \widehat{y}_i \leqq \sum_{i=1}^n \mathring{y}_i$, with equality if $\widehat{p}_y > 0$.

Therefore the First Welfare Theorem applies: if the utility functions and production functions satisfy the usual assumptions, then the equilibrium allocation will be Pareto efficient.

But Arrow's model also makes it clear that the Walrasian models's price-taking assumption for consumers is unrealistic here: for each of the distinct goods X_i there is only one person on the demand side of the market. The only person who cares about the good X_i is person i. It's clearly unrealistic to assume that any of the participants will take their own price (or Lindahl cost share) as given. This was Arrow's motivation for modeling things this way — to clarify this point.