Game Forms and Mechanism Design

Recall that a **game** is an n-tuple $(S_i, \pi_i)_{i=1}^n$, where

S_i is i's strategy or action set $(i = 1, \ldots, n)$,

$\pi_i : S_1 \times \cdots \times S_n \to \mathbb{R}$ is i's payoff function $(i = 1, \ldots, n)$.

A **game form** is a way to model the rules of a game, or an institution, independently of the players' utility functions over the game's outcomes. The notion of a game form is an important idea for *mechanism design* (also called *institution design* or *market design*).

Definition: Let X be a set of possible **outcomes**. A **game form** for X consists of

(1) n **action sets** $S_1, \ldots, S_n$, and

(2) an **outcome function** $\varphi : S_1 \times \cdots \times S_n \to X$.

Definition: Given an outcomes set X and

(1) a game form $(S_1, \ldots, S_n; \varphi)$ for X, and

(2) n utility functions $u_i : X \to \mathbb{R}$ over outcomes $(i = 1, \ldots, n)$,

the **associated game** or **induced game** is defined by the n action sets $S_1, \ldots, S_n$ and the n payoff functions

$$\widetilde{u}_i(s_1, \ldots, s_n) := u_i(\varphi(s_1, \ldots, s_n)),\ i = 1, \ldots, n.$$

In our public goods model, where x is the level at which the public good is provided and y_i is the number of dollars i spends on other goods, an outcome is an $(n+1)$-tuple $(x, y_1, \ldots, y_n) \in \mathbb{R}^{n+1}_+$, so our outcome set is $X = \mathbb{R}^{n+1}_+$. Assume that the cost of the public good is given by $C(x) = cx$, so marginal cost is c (for example, c is the price that's charged for each unit of the public good).

Example: The Voluntary Contributions Mechanism (VCM) for a public good.

The **VCM** institution, or game form, is defined by the following action sets and outcome function:

Actions: Each person i chooses a **contribution** m_i in the action set $\mathbb{R}_+$. Let $\mathbf{m} = (m_1, \ldots, m_n)$.

Outcome function:

$x = \pi(\mathbf{m}) = \frac{1}{c}\sum_1^n m_i$ (*i.e.*, x is whatever quantity the contributions $\sum_1^n m_i$ will buy);

$y_i = \mathring{y}_i - t_i$, where $t_i = \tau^i(\mathbf{m}) = m_i$ (*i.e.*, i's "tax" is simply his contribution, m_i) .

Thus, the outcome function is $\varphi(\mathbf{m}) = (\pi(\mathbf{m}), \mathring{y}_1 - \tau^1(\mathbf{m}), \ldots, \mathring{y}_n - \tau^n(\mathbf{m}))$.

The induced game is given by the utility functions $u^i(x, y_i),\ i = 1, \ldots, n$, so the payoff functions in the induced game are

$$\widetilde{u}^i(m_1, \ldots, m_n) := u^i(\pi(\mathbf{m}),\ \mathring{y}_i - \tau^i(\mathbf{m})) = u^i\Big(\frac{1}{c}\sum_{j=1}^n m_j,\ \mathring{y}_i - m_i\Big),\ i = 1, \ldots, n.$$

The Nash equilibrium of the VCM institution (*i.e.*, the NE of the associated game) is as follows:

The first-order marginal condition that characterizes individual i's choice of m_i is

$$\text{(FOMC)} \qquad \frac{\partial \widetilde{u}^i}{\partial m_i} \leqq 0 \quad \text{and} \quad \frac{\partial \widetilde{u}^i}{\partial m_i} = 0 \text{ if } m_i > 0.$$

We have

$$\frac{\partial \widetilde{u}^i}{\partial m_i} = \frac{\partial u^i}{\partial x_i}\frac{\partial \pi}{\partial m_i} + \frac{\partial u^i}{\partial y_i}\frac{\partial(-\tau^i)}{\partial m_i} = u^i_x \cdot \frac{1}{c} + u^i_y \cdot (-1) = \frac{1}{c}u^i_x - u^i_y.$$

Therefore

$$\frac{\partial \widetilde{u}^i}{\partial m_i} \leqq 0 \quad \text{if and only if} \quad \frac{u^i_x}{u^i_y} \leqq c.$$

Therefore the FOMC above, for individual i, can be written as

$$\frac{u^i_x}{u^i_y} \leqq c \quad \text{and} \quad \frac{u^i_x}{u^i_y} = c \text{ if } m_i > 0$$

$$\textit{i.e.,} \qquad MRS^i \leqq MC \quad \text{and} \quad MRS^i = MC \text{ if } m_i > 0.$$

Note that this is identical to the *market outcome* we obtained earlier, in which the public good is provided at a level that's less than the Pareto level: those who contribute are only those with the largest MRS^i; everyone else is a free rider; and *no one* will contribute if everyone has $MRS^i < MC$ when $x = 0$.

Mechanism Design: The mechanism design problem is to devise an outcome function φ for which the Nash equilibria (or some other specified solution) have one or more desirable properties — for example, an outcome function for which the Nash equilibria are Pareto efficient. For our simple public-goods model, the outcome function φ is the $(n+1)$-tuple of functions $(\pi, \tau^1, \ldots, \tau^n)$, so our mechanism design problem is to devise a provision function π and tax/transfer functions τ^i for each i for which the Nash equilibrium is Pareto efficient, or better yet, is a Lindahl equilibrium allocation.

The first institution/mechanism with Pareto efficient Nash equilibria was devised by Grove & Ledyard. The first mechanism with Lindahl Nash equilibria was devised by Leo Hurwicz.