Public Goods: Pareto Efficiency and Market Outcomes

A Motivating Example: Water Skiing vs. Sunbathing

Read the Introductory Notes for Microeconomics. Note that the Pareto marginal condition for two persons is shown to be *not* $MRS^1 = MRS^2 = MC$, but $MRS^1 + MRS^2 = MC$ instead. And since $MC = 0$ in the example, we obtained $MRS^1 + MRS^2 = 0$.

A Second Motivating Example: Mosquito Spray

The homeowners in a residential neighborhood are plagued by mosquitoes. The number of mosquitoes can be controlled by spraying. The mosquito spray is a public good because whatever amount is sprayed, this is the amount that is experienced (for good or bad) by all the homeowners: it's not possible to contain the spray so as to affect only the homeowner who purchases it. Let x denote the number of tankfuls of spray that are sprayed. For each $i \in N = \{1, \ldots, n\}$ let y_i denote household i's dollar expenditure on other goods, and let $u^i(x, y_i)$ be household i's utility function. An allocation is an $(n+1)$-tuple $(x, y_1, \ldots, y_n)$.

Pareto Efficiency:

We first derive the marginal conditions that characterize the Pareto allocations. The Pareto maximization problem is

$$\max_{x,(y^i)_1^n} \lambda_1 u^1(x, y_1) \quad \text{subject to } x, y_1, \ldots, y_n \geqq 0$$

$$\sum_{i=1}^n y_i + C(x) \leqq \mathring{y}, \qquad (\sigma) \qquad \text{(P-Max)}$$

$$u^i(x, y_i) \geqq u_i, \quad i = 2, ..., n. \qquad (\lambda_i)$$

The first-order marginal conditions for an interior solution are

$\exists\, \sigma \geqq 0$ and $\lambda_2, ..., \lambda_n \geqq 0$ such that

$$\lambda_1 u_x^1 + \lambda_2 u_x^2 + \ldots \lambda_n u_x^n = \sigma C'(x) \quad \text{and} \quad \lambda_i u_y^i = \sigma, \quad i = 1, ..., n. \qquad \text{(FOMC)}$$

Combining these first-order equations yields

$$\sum_{i=1}^n MRS^i = MC,$$

which is the **Samuelson Marginal Condition** for Pareto efficiency with a public good.

The Market Outcome:

In our mosquito-spray example, assume that there's a market in which firms provide mosquito-control spraying service at a price p per tankful of spray. Let's also assume that the homeowners are price-takers. In this case that's not enough to define the individual homeowner's decision problem: the amount of spray an individual wishes to purchase will be affected by how much the other homeowners purchase. What the individual homeowner cares about is the total amount of spray purchased by *everyone*, which we've denoted by x. Let ξ_i denote the amount of spray purchased by individual i; and let X_{-i} denote the total purchased by everyone else: $X_{-i} = \sum_{j \neq i} \xi_j$. Then $x = \sum_1^n \xi_j = X_{-i} + \xi_i$. Let's assume, then, that each individual i takes both the market price p *and* the total amount purchased by all the others, X_{-i}, as given.

The decision problem for each individual i is

$$\text{(U-max)} \qquad \max_{(\xi_i, y_i) \in \mathbb{R}^2_+} u^i(x, y_i) = u^i(X_{-i} + \xi_i, y_i) \text{ subject to } p\xi_i + y_i \leqq \mathring{y}_i,$$

or equivalently,

$$\max \ u^i(X_{-i} + \xi_i, \mathring{y}_i - p\xi_i) \ \text{ for } \ \xi_i \in [0, \mathring{y}_i/p].$$

The first-order marginal condition (assuming that $\xi_i < \mathring{y}_i/p$) is

$$MRS^i \leqq p \qquad \text{and} \qquad MRS^i = p \ \text{ if } \ \xi_i > 0.$$

The diagrams in Figure 1 depict the individual's decision problem. The total amount everyone else has purchased is X_{-i}. That's the smallest level of x individual i can obtain, by choosing $\xi_i = 0$. And it's the level he *will* choose unless his MRS^i at X_{-i} exceeds p. If his MRS^i does exceed p at X_{-i}, he will choose ξ_i (and therefore x) up to the level at which his $MRS^i = p$.

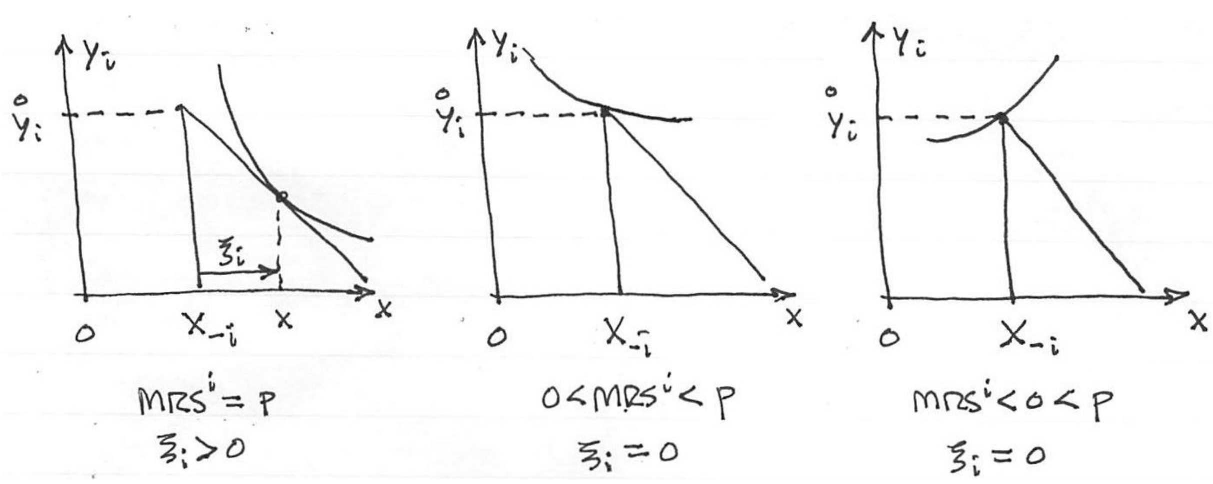

Figure 1

This leads naturally to the following definition of equilibrium:

Definition: Let p be the price at which a public good is provided. A **public-good price-taking Nash equilibrium** at price p is an n-tuple $(\xi_1, \ldots, \xi_n) \in \mathbb{R}^n_+$ that satisfies (U-max) for each $i = 1, \ldots, n$.

Clearly, if an equilibrium has $x > 0$, then some individual $h \in N$ must satisfy $\xi_h > 0$ and therefore $MRS^h = p$. Each i whose MRS^i is less than p will not purchase any of the public good (*i.e.*, $\xi_i = 0$ for each such i), but some or all of these individuals' marginal rates of substitution — their marginal values for the public good — may nevertheless be well above zero. Consequently we would have $\sum_1^n MRS^i > p$, and indeed the sum will often be substantially larger than p.

For the n consumers of the public good, note that the marginal cost *to them* of an additional unit of the good is its price p. Thus, to the n consumers, a market equilibrium typically satisfies the inequality $\sum_1^n MRS^i > MC$ — the equilibrium is not Pareto efficient, because the equilibrium level of x is too low. And if $\sum_1^n MRS^i$ is *substantially* larger than p, then the equilibrium x may be substantially less than Pareto efficiency would require, as in the following examples.

Example 1: There are five homeowners: $N = \{1, 2, 3, 4, 5\}$. Their utility functions are all of the form $u(x, y_i) = y_i - \frac{1}{2}(\alpha_i - x)^2$, where x denotes the level at which a public good is provided, and y_i denotes the amount of money homeowner i has available to spend on other goods. The values of their preference parameters α_i are

$$\alpha_1 = 30, \qquad \alpha_2 = 27, \qquad \alpha_3 = 24, \qquad \alpha_4 = 21, \qquad \alpha_5 = 18,$$

and their MRS functions are therefore

$$MRS^1 = 30 - x, \quad MRS^2 = 27 - x, \quad MRS^3 = 24 - x, \quad MRS^4 = 21 - x, \quad MRS^5 = 18 - x.$$

The firms that produce the public good all charge a per-unit price of p dollars; p is therefore the marginal cost to the homeowners for each unit of x. Suppose $p = \$40$. Because the utility functions are quasilinear, there's a unique Pareto level of the public good, namely $x = 16$:

$$\Sigma MRS^i = 120 - 5x \ \text{ and } \ MC = 40, \quad \text{therefore } \Sigma MRS^i = MC \text{ at } x = 16.$$

And because each $i \in N$ has $MRS^i < p$ at every $x \in \mathbb{R}_+$, there is also a unique equilibrium:

$$\xi_i = 0 \text{ for all } i, \quad \text{and therefore } x = 0.$$

None of the public good is purchased, despite the fact that the Pareto level is $x = 16$ and despite the fact that when $x = 0$, the marginal social value of the public good, ΣMRS^i, is 120, which far exceeds the marginal cost (*i.e.*, the \$40 price) of each unit of x. Consumer surplus at the Pareto provision level, $x = 16$, is \$640, all of which is foregone at the equilibrium.

Example 2: With the same five consumers as in Example 1, suppose $p = \$20$. In this case the unique Pareto level of x is $x = 20$. There is again a unique equilibrium: $(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5) = (10, 0, 0, 0, 0)$ and therefore $x = 10$. Note that at the equilibrium outcome, $MRS^1 = \$20 = p$ and for all other i, $0 < MRS^i < p$, as in the middle diagram in Figure 1. In everyday, nontechnical language, the other four consumers would be said to be "free riding" on Consumer 1, and would be referred to as "free riders:" they're purchasing none of the public good while receiving positive benefit from Consumer 1's purchase. We have $\Sigma MRS^i = 70$, which is substantially larger than the \$20 price. Consumer surplus at the Pareto provision level, $x = 20$, is \$1000; consumer surplus at the equilibrium is \$750.

Example 3: Suppose the price is $p = \$20$ as in Example 2, but that the preference parameters in Examples 1 and 2 are changed to

$$\alpha_1 = \alpha_2 = \alpha_3 = 30, \qquad \alpha_4 = 25, \qquad \alpha_5 = 5,$$

so that the MRS functions are now

$$MRS^1 = MRS^2 = MRS^3 = 30 - x, \quad MRS^4 = 25 - x, \quad MRS^5 = 5 - x.$$

We still have $\Sigma MRS^i = 120 - 5x$, so the Pareto level of x is still $x = 20$, as in Example 2. But now there are multiple equilibria: the equilibria are all the $(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5)$ that satisfy $\xi_1 + \xi_2 + \xi_3 = 10$ and $\xi_4 = \xi_5 = 0$. In each of the equilibria we have $MRS^i = p = 20$ for $i = 1, 2, 3$, and we have $MRS^4 = 15 < p$ and $MRS^5 = -5 < p$. Note that Consumer 5 is not a free rider here: her consumer surplus is zero. She receives some surplus on the first five units of x, which is just offset by the negative consumer surplus she receives from the next five units. Her decision problem corresponds to the rightmost diagram in Figure 1.

Example 4: In Example 3, change α_4 to 28 and α_5 to 2. The Pareto level of x and the equilibria are unchanged, but now $MRS^4 = 18$ and $MRS^5 = -8$ at the equilibria. Now Consumer 5 "suffers damages" at the equilibrium: her consumer surplus is $-\$30$. She is worse off than if none of the public good were provided. For example, think of a homeowner who experiences respiratory difficulties from mosquito spray if it's provided at a level greater than $x = 2$.

An Alternative Institution:

Suppose the homeowners form a homeowners association (HOA) to deal with their mosquito problem: the HOA will accept voluntary contributions from the homeowners, and will then use the total contributions to purchase as much mosquito spray as the contributions will buy. More formally, each homeowner $i \in N$ chooses to contribute a dollar amount $t_i \in \mathbb{R}_+$ to the mosquito fund. The total amount contributed is $\Sigma_{i \in N} t_i$. Then the HOA purchases $x = \frac{1}{p}\Sigma_{i \in N} t_i$ tanks of spray, where p is the price per tank.

We'll assume that each homeowner takes the total of all the *others'* contributions as given, and chooses his own contribution t_i to maximize his utility. Let T_{-i} denote the total of the others' contributions: $T_{-i} = \Sigma_{j \neq i} t_j$. The individual's maximization problem is

$$(*) \qquad \max_{t_i \in \mathbb{R}_+} u^i(x, y_i), \quad \text{where } x = \frac{1}{p}(T_{-i} + t_i) \text{ and } y_i = \mathring{y} - t_i.$$

We define an equilibrium as follows:

Definition: A **voluntary contributions equilibrium** for a public good with price $p \in \mathbb{R}_{++}$ is an n-tuple $(t_1, \ldots, t_n) \in \mathbb{R}^n_+$ in which, for each $i \in N$, t_i is a solution of $(*)$.

The first-order marginal condition for each individual's maximization problem $(*)$ is

$$\frac{1}{p}u^i_x - u^i_y \leqq 0 \;\text{ and }\; \frac{1}{p}u^i_x - u^i_y = 0 \text{ if } t_i > 0,$$

i.e.,

$$MRS^i \leqq p \;\text{ and }\; MRS^i = p \text{ if } t_i > 0.$$

This is the same marginal condition as in the individual-purchases institution we analyzed above. Therefore a voluntary contributions equilibrium and a price-taking equilibrium are identical, with the individual actions t_i and ξ_i related by the equations $t_i = p\xi_i$ for each $i \in N$. In Example 1 we have $t_i = 0$ for each $i \in N$, and $x = 0$. In Example 2 we have $t_1 = \$200$ and $t_i = 0$ for $i = 2, 3, 4, 5$, and $x = 10$. In Examples 3 and 4 we have $t_1 + t_2 + t_3 = \$200$ and $t_4 = t_5 = 0$, and $x = 10$.