Econ 501B Final Exam Solutions
Fall 2010

(1.) $u^A(x_0, x_H, x_L) = x_0 + 4\sqrt{x_H} + \sqrt{x_L}$ $\quad (\mathring{x}_0^A, \mathring{x}_H^A, \mathring{x}_L^A) = (10, 10, 8)$

$u^B(x_0, x_H, x_L) = x_0 + 3\sqrt{x_H} + 2\sqrt{x_L}$ $\quad (\mathring{x}_0^B, \mathring{x}_H^B, \mathring{x}_L^B) = (10, 15, 12)$

$$MRS_H^A = \frac{2}{\sqrt{x_H^A}}, \quad MRS_L^A = \frac{1}{2\sqrt{x_L^A}}$$

$$MRS_H^B = \frac{3}{2\sqrt{x_H^B}}, \quad MRS_L^B = \frac{1}{\sqrt{x_L^B}}.$$

(a) $MRS_H^A = MRS_H^B$: $\frac{2}{\sqrt{x_H^A}} = \frac{3}{2\sqrt{x_H^B}}$; i.e., $\frac{4}{x_H^A} = \frac{9}{4x_H^B}$

$$9x_H^A = 16x_H^B = 16(25 - x_H^A) = (16)(25) - 16x_H^A$$

i.e., $25x_H^A = (25)(16)$; $\therefore \boxed{x_H^A = 16, \ x_H^B = 9}$

$\therefore MRS_H^A = MRS_H^B = \frac{1}{2}$.

$MRS_L^A = MRS_L^B$: $\frac{1}{2\sqrt{x_L^A}} = \frac{1}{\sqrt{x_L^B}}$; i.e., $\frac{1}{4x_L^A} = \frac{1}{x_L^B}$

$4x_L^A = x_L^B = 20 - x_L^A$; $\quad 5x_L^A = 20$; $\therefore \boxed{x_L^A = 4, \ x_L^B = 16}$

$\therefore MRS_L^A = MRS_L^B = \frac{1}{4}$.

x_0^A, x_0^B need only satisfy $\boxed{x_0^A + x_0^B = 20}$.

(b) Because the A-D equilibrium is Pareto optimal (1st Welfare Theorem), we have $x_H^A, x_H^B, x_L^A, x_L^B$ as above, and $p_H = \frac{1}{2}$, $p_L = \frac{1}{4}$ (assuming $p_0 \equiv 1$).

$\therefore p_H(x_H^A - \mathring{x}_H^A) + p_L(x_L^A - \mathring{x}_L^A) = \frac{1}{2}(16 - 10) + \frac{1}{4}(4 - 8)$

$= \frac{1}{2}(6) + \frac{1}{4}(-4) = 3 - 1 = 2$;

$\therefore \boxed{x_0^A = \mathring{x}_0^A - 2 = 8; \ x_0^B = \mathring{x}_0^B + 2 = 12}$

(c) $\frac{1}{1+r} = p_H + p_L = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$; i.e., $1+r = \frac{4}{3}$, $\boxed{r = \frac{1}{3} = 33\frac{1}{3}\%}$

(d) $d_\gamma = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ AND $d_\delta = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

ARROW'S SECURITIES-PRICING FORMULA YIELDS PRICES

$q_\gamma = 3p_H + (-2)p_L = (3)(\frac{1}{2}) - (2)(\frac{1}{4}) = \frac{3}{2} - \frac{1}{2} = 1$

$q_\delta = 0p_H + 1p_L = \frac{1}{4}$. $\boxed{(q_\gamma, q_\delta) = (1, \frac{1}{4})}$

ALAN CAN OBTAIN HIS A-D CONSUMPTION OF $(x_H^A, x_L^A) = (\mathring{x}_H^A + 6, \mathring{x}_L^A - 4)$ VIA HOLDING $\boxed{(y_\gamma^A, y_\delta^A) = (2, 0)}$ AT A COST OF $2q_\gamma + 0q_\delta = (2)(1) = 2$, SO $x_0^A = \mathring{x}_0^A - 2 = 8$.

BEN WILL ACHIEVE $(x_H^B, x_L^B) = (\mathring{x}_H^B - 6, \mathring{x}_L^B + 4)$ VIA $\boxed{(y_\gamma^B, y_\delta^B) = (-2, 0)}$, AND $x_0^B = \mathring{x}_0^B - (-2) = 12$.

(e) WITH ONLY γ AVAILABLE, THE MARKET EQUILIBRIUM WILL STILL BE THE ARROW-DEBREU ALLOCATION (AND $\therefore$ PARETO OPTIMAL), BECAUSE BOTH ALAN AND BEN WILL CHOOSE y_γ^A AND y_γ^B THAT YIELD THEIR A-D CONSUMPTIONS. (THEY CHOSE $y_\delta^A = y_\delta^B = 0$ WHEN δ WAS ALSO AVAILABLE.) BUT THIS IS NOT AT ALL ROBUST: A SLIGHT CHANGE IN THE PREFERENCES OR ENDOWMENTS WILL (GENERICALLY) CHANGE THE A-D EQUILIBRIUM CONSUMPTIONS TO $(\hat{x}_H^i, \hat{x}_L^i)$ THAT ARE <u>NOT</u> A MULTIPLE OF d_γ, SO THAT (x_H^i, x_L^i) <u>CANNOT</u> BE ACHIEVED VIA JUST THE SINGLE SECURITY γ.

(2) (a) $\max \lambda_A u^A(x_{AM}, x_{AW}, x_{AO})$ s.t. $z \geqq 0$ AND ALL $x_{ik} \geqq 0$, AND

$$x_{AM} + x_{BM} \leqq f(z) \qquad (\sigma_M) \leftarrow \text{LAGRANGE MULTIPLIERS}$$
$$x_{AW} + x_{BW} \leqq f(z) \qquad (\sigma_W)$$
$$u^B(x_{BM}, x_{BW}, x_{BO}) \geqq \bar{u}^B \qquad (\lambda_B)$$

(b)
$$\left.\begin{array}{l} x_{iO}: \lambda_i u_{iO} = \sigma_O \quad [i=A,B] \\ x_{ik}: \lambda_i u_{ik} = \sigma_k \quad [i=A,B,\ k=M,W] \end{array}\right\} \frac{u_{ik}}{u_{iO}} = \frac{\sigma_k}{\sigma_O}, \text{ i.e. } MRS_{ik} = \frac{\sigma_k}{\sigma_O}$$
$$z: \quad 0 = \sigma_O - (\sigma_M + \sigma_W) f'(z); \text{ i.e., } \frac{1}{f'(z)} = \frac{\sigma_M}{\sigma_O} + \frac{\sigma_W}{\sigma_O}$$

IN ECONOMIC TERMS: $MRS_{AM} = MRS_{BM}$, $MRS_{AW} = MRS_{BW}$, AND $MRS_{iM} + MRS_{iW} = \frac{1}{f'(z)}$ $[i=A,B]$, AND $\frac{1}{f'(z)}$ IS THE REAL MARGINAL COST OF A COMBINED ONE-UNIT INCREASE IN BOTH MUTTON AND WOOL.

(c) AT PRICES p_O, p_M, p_W, PROFIT IS DEFINED BY

$$\pi(z) := (p_M + p_W)\frac{1}{c} z - p_O z = \left[(p_M + p_W)\frac{1}{c} - p_O\right] z$$

AT INPUT LEVEL z. IF $\pi(z) > 0$, THEN

$$\pi(\lambda z) = \left[(p_M + p_W)\frac{1}{c} - p_O\right]\lambda z = \lambda \pi(z) > \pi(z)$$

IF $\lambda > 1$; i.e., INCREASING PRODUCTION WILL INCREASE PROFIT, SO z IS NOT A PROFIT-MAXIMIZING CHOICE. IF $\pi(z) < 0$, THEN $\pi(z) < \pi(0) = 0$, SO AGAIN z IS NOT A PROFIT-MAXIMIZING CHOICE. THEREFORE $\pi(z) = 0$ AT ANY PROFIT-MAXIMIZING CHOICE OF INPUT LEVEL z, IN ANY EQUILIBRIUM. THEREFORE WE ALSO HAVE $(p_M + p_W)\frac{1}{z} = p_O$ IN ANY EQUILIBRIUM. NOTE: THIS RESULT IS NOT BECAUSE THE FIRM CHOOSES z TO EQUATE MR AND MC; MR AND MC ARE EXOGENOUS TO THE FIRM.

(d) CONSUMERS: $\max u(x_m, x_w, x_o)$ s.t. $p_o x_o + p_m x_m + p_w x_w \leqq \omega$.

FOMC:
$$\left.\begin{aligned} x_o &: u_o = \lambda p_o \\ x_m &: u_m = \lambda p_m \\ x_w &: u_w = \lambda p_w \end{aligned}\right\} \quad \frac{u_m}{u_o} = \frac{p_m}{p_o} \text{ AND } \frac{u_w}{u_o} = \frac{p_w}{p_o}$$

i.e., $MRS_{im} = p_m$ AND $MRS_{iw} = p_w$

(BECAUSE $p_o \equiv 1$).

FIRM(S): AS ABOVE, WE MUST HAVE $\pi(z) = 0$ IN AN EQUILIBRIUM, AND $\therefore$ $(p_m + p_w)\frac{1}{c} = p_o$; ↖ BECAUSE FIRMS ARE PRICE-TAKERS

i.e., $p_m + p_w = c = \dfrac{1}{f'(z)}$.

(e) $MRS_{im} = p_m$ & $MRS_{iw} = p_w$ $[i = A, B]$ YIELDS

$$MRS_{Am} = MRS_{Bm} \text{ AND } MRS_{Aw} = MRS_{Bw},$$

AND COMBINED WITH $p_m + p_w = \dfrac{1}{f'(z)}$,

WE HAVE $\quad MRS_{im} + MRS_{iw} = \dfrac{1}{f'(z)} \quad [i = A, B]$

— EXACTLY THE FOMC FOR PARETO ALLOCATIONS IN (b).

~~(f) HERE WE ARE TECHNOLOGICALLY CONSTRAINED TO HAVE~~

(f) THIS QUESTION IS ACTUALLY MISTAKEN: THE FOMC IN (d) ARE NOT THE SAME AS WITH ONLY A CREDIT MARKET, WHERE WE DO NOT HAVE THE ANALOGUES OF $MRS_{im} = p_m$ AND $MRS_{iw} = p_w$ — WE HAVE ONLY $MRS_{im} + MRS_{iw} = \dfrac{1}{1+r}$.

(3) (a) MARKET EQUILIBRIUM: $p_x = p_y$ (=1, SAY)

$(x_A, y_A) = (x_B, y_B) = (18, 18)$.

TO VERIFY THAT THIS IS AN EQUILIBRIUM:

(U-MAX) EACH (x_i, y_i) MAX'S $u_i(x_i, y_i)$?

$MRS_i = \frac{p_x}{p_y}$? $\quad MRS_i = \frac{18}{18} = 1$ AND $\frac{p_x}{p_y} = 1$. OK ✓

ON THE BUDGET CONSTRAINT? $p_x x_i + p_y y_i = 18 + 18 = 36$

$= p_x \mathring{x}_i + p_y \mathring{y}_i$ OK ✓

(MKT-CLEARING) $x_A + x_B = \mathring{x}$ AND $y_A + y_B = \mathring{y}$?

$18 + 18 = 36$ IN BOTH CASES. OK ✓

(b) PARETO REQUIRES $MRS_A = MRS_B$ — i.e., $\sqrt{\frac{y_A}{x_A}} = \sqrt{\frac{y_B}{x_B}}$.

EQUIVALENTLY, $\frac{y_A}{x_A} = \frac{y_B}{x_B} = r$, SAY.

$\therefore y_A = r x_A$ AND $y_B = r x_B$; $\therefore y_A + y_B = r(x_A + x_B)$;

BUT ALSO $x_A + x_B = \mathring{x}$ AND $y_A + y_B = \mathring{y}$, SO $r = \frac{\mathring{y}}{\mathring{x}} = \frac{36}{36} = 1$.

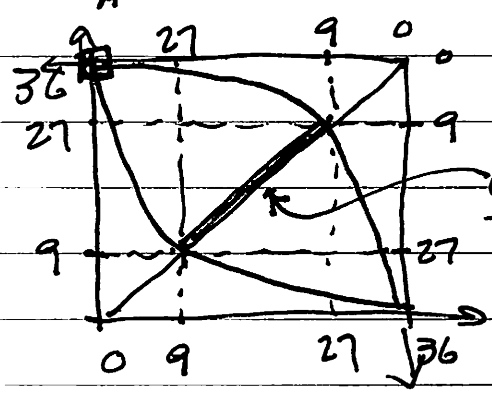
$\therefore y_A = x_A$ AND $y_B = x_B$; $u_i = \sqrt{x_i} + \sqrt{y_i} = 2\sqrt{x_i}$

$\therefore (u_i)^2 = 4x_i$ AND $u_A^2 + u_B^2 = 4(x_A + x_B) = 4\mathring{x} = 144$.

THE UTILITY FRONTIER IS $\boxed{u_A^2 + u_B^2 = 144}$

(c) PARETO HERE REQUIRES $y_i = x_i$ $(i = A, B)$,

$\therefore u_i = 2\sqrt{x_i}$. WE ALSO HAVE $\mathring{u}_A = \mathring{u}_B = \sqrt{36} = 6$.



$\therefore$ THE CORE THEREFORE CONSISTS OF THE ALLOCATIONS THAT SATISFY $y_A = x_A$, $y_B = x_B$,

$x_A + x_B = y_A + y_B = 36$, AND

$u_A = 2\sqrt{x_A} \geqq 6$, i.e., $x_A \geqq 9$, AND SIMILARLY, $x_B \geqq 9$.

(d) MARKET EQUILIBRIUM: $p_x = p_y$ AND $(x_A, y_A) = (x_B, y_B) = 18$ AND $(x_c, y_c) = (12, 12)$.

VERIFICATION IS ALMOST THE SAME AS IN (a):

A AND B ARE OBVIOUSLY STILL MAXIMIZING THEIR UTILITY (SAME PRICES, SAME BUNDLES);

C IS ALSO MAXIMIZING UTILITY:

$$MRS_c = 1 = \frac{p_x}{p_y} \quad \text{AND} \quad p_x x_c + p_y y_c = p_x \mathring{x}_c + p_y \mathring{y}_c = 24 p_x = 24 p_y.$$

MARKETS CLEAR:

$$x_A + x_B + x_c = 48 = \mathring{x} \quad \text{AND} \quad y_A + y_B + y_c = 48 = \mathring{y}.$$

(e) THE EQUAL-MRS CONDITION YIELDS $y_i = x_i$ $(i = A, B, C)$. WE CAN THEREFORE REPRESENT ALL THE POTENTIAL CORE ALLOCATIONS BY JUST THE ALLOCATION OF THE X-GOOD: THE ALLOCATIONS (x_A, x_B, x_c) THAT SATISFY $x_A + x_B + x_c = 48$ AND WHICH NO ONE-PERSON OR TWO-PERSON COALITION CAN IMPROVE UPON. THESE CAN BE DEPICTED IN THE SIMPLEX $\{(x_A, x_B, x_c) \mid x_A + x_B + x_c = 48\}$ IN $\mathbb{R}^3_+$, AS SHOWN ON A SUBSEQUENT PAGE. THE NO-IMPROVEMENT CONDITIONS ARE AS FOLLOWS:

$S = \{A\}$ OR $\{B\}$: $u_i \geqq \mathring{u}_i = \sqrt{36} = 6$; AND SINCE $x_i = y_i$, WE MUST HAVE $u_i = 2\sqrt{x_i} \geqq 6$ — i.e., $4x_i \geqq 36$, SO $x_i \geqq 9$. THIS GIVES US THE INEQUALITIES $x_A \geqq 9$ AND $x_B \geqq 9$.

$S = \{C\}$: $u_c \geqq \mathring{u}_c = 2\sqrt{12}$; $\therefore 2\sqrt{x_c} \geqq 2\sqrt{12}$, i.e., $x_c \geqq 12$.

$S = \{A, B\}$: $\mathring{x}_S = \mathring{y}_S = 36$; $\therefore$ PARETO ALLOCATIONS OF $(\mathring{x}_S, \mathring{y}_S)$ TO A AND B MUST SATISFY $x_A = y_A$, $x_B = y_B$, AND $x_A + x_B = 36$.

THEREFORE $u_A = 2\sqrt{x_A}$ AND $u_B = 2\sqrt{x_B}$, AND $u_A^2 + u_B^2 = 4(x_A + x_B)$, SO $\{A, B\}$ CAN ACHIEVE $u_A^2 + u_B^2 = 4(\mathring{x}_A + \mathring{x}_B)$. SINCE CORE ALLOCATIONS ALSO SATISFY $u_i = 2\sqrt{x_i}$, THEY MUST SATISFY $x_A + x_B \geqq \mathring{x}_A + \mathring{x}_B = 36$ TO ENSURE THAT $\{A, B\}$ CANNOT IMPROVE.

NOTE THAT COMBINING THE NO-IMPROVEMENT CONDITIONS FOR $\{A, B\}$ AND $\{C\}$, WE HAVE $x_A + x_B \geqq 36$ AND $x_C \geqq 12$; $\therefore$ $x_A + x_B = 36$ AND $x_C = 12$, SINCE WE ALSO HAVE $x_A + x_B + x_C = 48$.

$S = \{A, C\}$: $(\mathring{x}_S, \mathring{y}_S) = (12, 48)$; $\therefore$ THE PARETO ALLOCATIONS OF $(\mathring{x}_S, \mathring{y}_S)$ TO A AND C MUST SATISFY $y_A = 4x_A$ AND $y_C = 4x_C$ AS WELL AS $x_A + x_C = 12$.

$\therefore$ $u_i = \sqrt{x_i} + \sqrt{4x_i} = 3\sqrt{x_i}$ AND $u_i^2 = 9x_i$ $(i = A, C)$

$\therefore$ S CAN ACHIEVE $u_A^2 + u_C^2 = 9(x_A + x_C) = 9(\mathring{x}_A + \mathring{x}_C) = 108$.

SINCE CORE ALLOCATIONS HAVE $x_i = y_i$ (ALL i), THEY YIELD

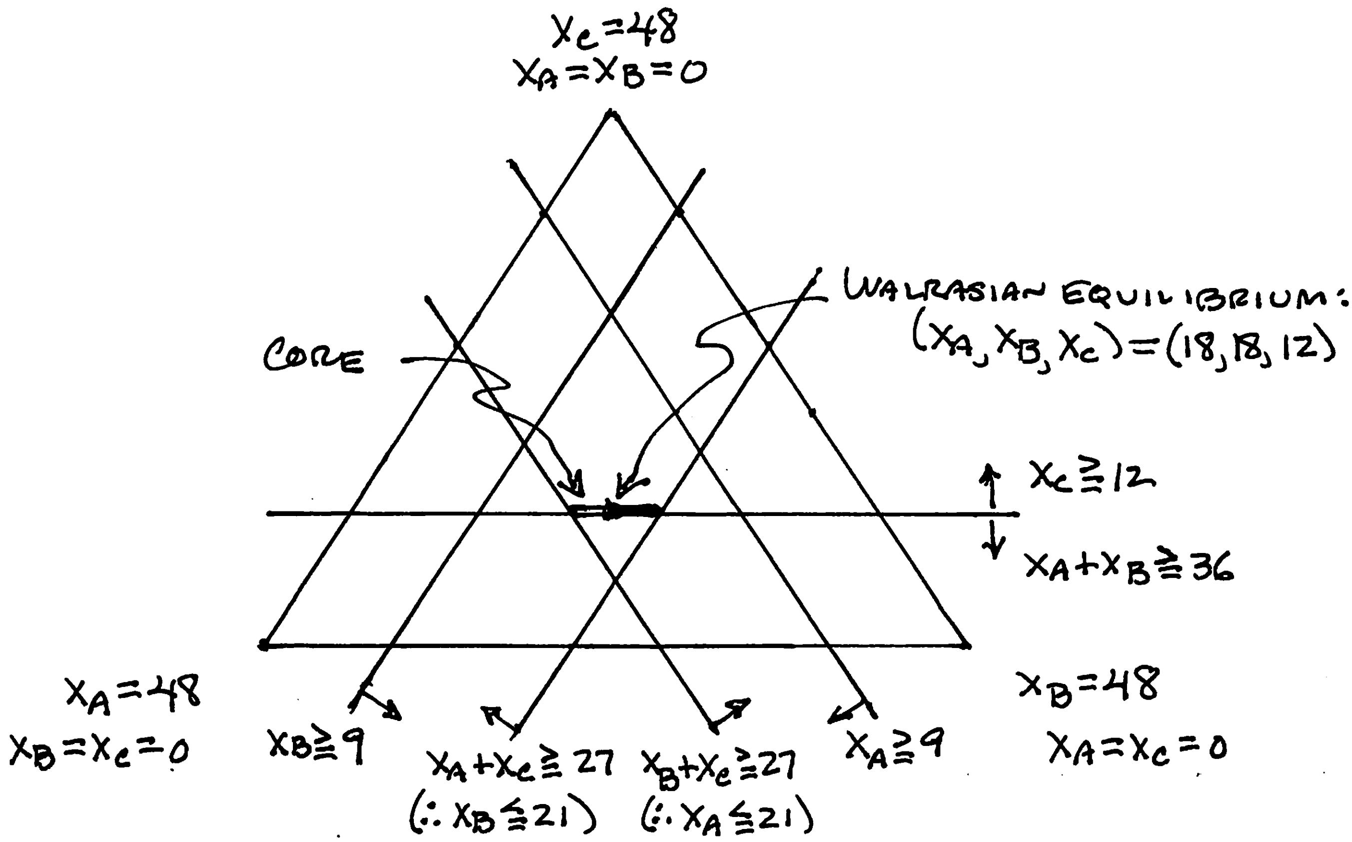
$$u_A^2 + u_C^2 = (2\sqrt{x_A})^2 + (2\sqrt{x_C})^2 = 4x_A + 4x_C = 4(x_A + x_C).$$

IN ORDER THAT S CAN'T IMPROVE, WE MUST HAVE

$4(x_A + x_C) \geqq 108$ — i.e., $x_A + x_C \geqq 27$ (AND $\therefore$ $x_B \leqq 21$).

$S = \{B, C\}$ IS SYMMETRIC TO $\{A, C\}$: THE NO-IMPROVEMENT CONDITION IS $x_B + x_C \geqq 27$ (AND $\therefore$ $x_A \leqq 21$).

ALL THESE NO-IMPROVEMENT INEQUALITY CONDITIONS ARE DEPICTED IN THE SIMPLEX ON THE FOLLOWING PAGE.



THE CORE ALLOCATIONS: $x_A + x_B + x_C = 48$

[NOTE: $x_A \geqq 9$ AND $x_B \geqq 9$ ARE NOT BINDING.]

$x_A = y_A,\ x_B = y_B,\ x_C = y_C$

$x_A + x_B = 36,\ x_C = 12$

$x_A + x_C \geqq 27,\ x_B + x_C \geqq 27$

THESE CAN BE SIMPLIFIED TO $x_C = 12,\ x_A + x_B = 36$

$15 \leqq x_A \leqq 21,\ 15 \leqq x_B \leqq 21.$

ADDING CAY HAS NARROWED THE RANGE OF CORE ALLOCATIONS TO AMY AND BEV, FROM $9 \leqq x_i \leqq 27$ WITHOUT CAY TO $15 \leqq x_i \leqq 21$ WITH CAY. THE CORE ALLOCATIONS TO AMY AND BEV ARE OTHERWISE UNCHANGED: THEY STILL DIVIDE 36 UNITS OF BOTH GOODS, AND THEY STILL RECEIVE BUNDLES THAT SATISFY $x_A = y_A$ AND $x_B = y_B$.