ECON 501B FINAL EXAM SOLUTIONS

FALL 2011

① (a) <u>PROOF:</u> SUPPOSE $\hat{x}$ IS NOT A SOLUTION OF THE MAXIMIZATION PROBLEM — i.e., THERE IS AN $\tilde{x} \in X$ THAT SATISFIES $u^i(\tilde{x}) \geqq u^i(\hat{x})$ FOR $i = 2, \ldots, n$ AND THAT ALSO SATISFIES $u^1(\tilde{x}) > u^1(\hat{x})$. THEN $\tilde{x}$ IS A PARETO IMPROVEMENT ON $\hat{x}$, SO $\hat{x}$ IS NOT PARETO.

(b) <u>PROOF:</u> SUPPOSE $\hat{x}$ IS NOT PARETO EFFICIENT — i.e., SOME $\tilde{x} \in X$ IS A PARETO IMPROVEMENT ON $\hat{x}$, WHICH MEANS THAT $u^i(\tilde{x}) \geqq u^i(\hat{x})$ FOR ALL i AND ALSO $u^k(\tilde{x}) > u^k(\hat{x})$ FOR SOME k. THEN FOR ANY $\alpha_1, \ldots, \alpha_n > 0$ WE HAVE $\sum_{i=1}^n \alpha_i u^i(\tilde{x}) > \sum_{i=1}^n \alpha_i u^i(\hat{x})$, SO THERE ARE NO VALUES OF $\alpha_1, \ldots, \alpha_n$ FOR WHICH $\hat{x}$ MAXIMIZES $W(x) := \sum_{i=1}^n \alpha_i w_i(x)$.

(c) AS IN FIGURE 1, LET $(\mathring{x}_1, \mathring{x}_2) = (4,4)$; $\hat{x} = ((2,2),(2,2))$; $u^1(x) = x_{11}x_{12}$; AND

$$u^2(x) = \begin{cases} x_{21}x_{22}, & \text{IF } x_{21}x_{22} \leqq 1 \\ 1, & \text{IF } 1 \leqq x_{21}x_{22} \leqq 9 \\ x_{21}x_{22} - 8, & \text{IF } x_{21}x_{22} \geqq 9 \end{cases}$$

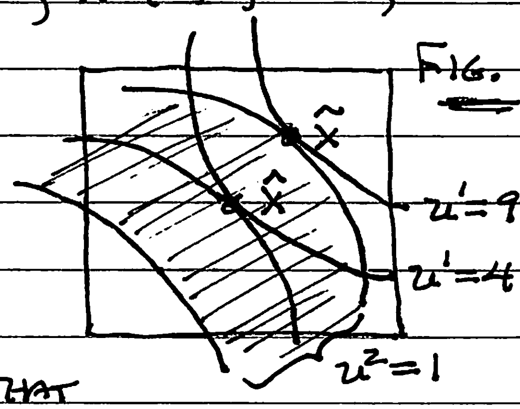

FIG. 1

THEN $\hat{x}$ IS A SOLUTION OF THE MAXIMIZATION PROBLEM. NOTE THAT $u^2(\hat{x}) = 1$ AND $\hat{x}_{21}\hat{x}_{22} = 4$. BUT $\tilde{x} = ((3,3),(1,1))$ ALSO SATISFIES $u^2(\tilde{x}) \geqq u^2(\hat{x})$ — i.e., $u^2(\tilde{x}) \geqq 1$ — AND $u^1(\tilde{x}) = 9 > u^1(\hat{x})$. THEREFORE $\tilde{x}$ IS A PARETO IMPROVEMENT ON $\hat{x}$, SO $\hat{x}$ IS NOT PARETO EFFICIENT.

(d) LET $u^1(x) = x_{11}x_{12}$, $u^2(x) = x_{21}x_{22}$, AND $(\mathring{x}_1, \mathring{x}_2) = (4,4)$. THEN x IS PARETO IF AND ONLY IF x IS ON THE DIAGONAL OF THE EDGEWORTH BOX — i.e., $x_{11} = x_{12}$ AND $x_{21} = x_{22}$. THAT IS, x IS PARETO IF AND ONLY IF IT HAS THE FORM $x = ((\xi,\xi),(4-\xi,4-\xi))$. FOR $\alpha_1, \alpha_2 > 0$ WE HAVE $W(x) = \alpha_1\xi^2 + \alpha_2(4-\xi)^2$ FOR THE PARETO ALLOCATIONS x; i.e.,

$$W(x) = \alpha_1\xi^2 + 16\alpha_2 - 8\alpha_2\xi + \alpha_2\xi^2$$

$$\frac{\partial W}{\partial \xi} = 2\alpha_1\xi - 8\alpha_2 + 2\alpha_2\xi = 2(\alpha_1+\alpha_2)\xi - 8\alpha_2$$

$$\frac{\partial^2 W}{\partial \xi^2} = 2(\alpha_1 + \alpha_2) > 0,$$

SO W IS STRICTLY CONVEX IN ξ AND W IS THEREFORE MAXIMIZED ONLY WHEN ξ IS AT AN ENDPOINT — i.e., $\xi = 0$ OR $\xi = 4$. CONSEQUENTLY, FOR ALL VALUES OF α_1 AND α_2, W IS NEVER MAXIMIZED AT ANY INTERIOR PARETO ALLOCATIONS, BUT ONLY AT THE CORNERS OF THE EDGEWORTH BOX: $((0,0),(4,4))$ AND $((4,4),(0,0))$.

(2) $u^A(x_{A0}, x_{AH}, x_{AL}) = x_{A0} + 30\log x_{AH} + 15\log x_{AL}$ $\quad \mathring{x}_{AS} = 30, \forall s$

$u^B(x_{B0}, x_{BH}, x_{BL}) = x_{B0} + 15\log x_{BH} + 15\log x_{BL}$ $\quad \mathring{x}_{BS} = 60, \forall s$

$\mathring{x}_S = 90, \forall s$

$$MRS_H^A = \frac{30}{x_H^A} \qquad MRS_L^A = \frac{15}{x_L^A}$$

$$MRS_H^B = \frac{15}{x_H^B} \qquad MRS_L^B = \frac{15}{x_L^B}$$

(a) $MRS_H^A = MRS_H^B$: $\frac{30}{x_{AH}} = \frac{15}{x_{BH}}$; $30x_{BH} = 15x_{AH}$; $x_{AH} = 2x_{BH}$

$\therefore x_{AH} = 60,\ x_{BH} = 30$

$MRS_L^A = MRS_L^B$: $\frac{15}{x_L^A} = \frac{15}{x_L^B}$; $x_{AL} = x_{BL}$

$\therefore x_{AL} = 45,\ x_{BL} = 45$

$\therefore MRS_H^A = MRS_H^B = \frac{1}{2},\ MRS_L^A = MRS_L^B = \frac{1}{3}$.

ONLY RESTRICTION ON x_{A0}, x_{B0} IS $x_{A0} + x_{B0} = 90$.

(b) $P_H = MRS_H^i = \frac{1}{2},\ P_L = MRS_L^i = \frac{1}{3}$.

(c) IN EQUIL'M: $MRS_H^i + MRS_L^i = \frac{1}{1+r},\ i = A, B,$

$\therefore MRS_H^A + MRS_L^A = MRS_H^B + MRS_L^B$

i.e., $\frac{30}{30+z_A} + \frac{15}{30+z_A} = \frac{15}{60+z_B} + \frac{15}{60+z_B}$

i.e., $\frac{45}{30+z_A} = \frac{30}{60+z_B}$; i.e., $\frac{3}{30+z_A} = \frac{2}{60+z_B}$

i.e., $180 + 3z_B = 60 + 2z_A$

IN EQUIL'M, $z_A + z_B = 0$, so $180 - 3z_A = 60 + 2z_A$

i.e., $5z_A = 120$, so $z_A = 24,\ z_B = -24$

$\therefore x_{AH} = x_{AL} = 54$ AND $x_{BH} = x_{BL} = 36$.

THIS YIELDS $MRS^A_H + MRS^A_L = \frac{30}{54} + \frac{15}{54} = \frac{45}{54} = \frac{5}{6}$

AND $MRS^B_H + MRS^B_L = \frac{15}{36} + \frac{15}{36} = \frac{30}{36} = \frac{5}{6}$

$\therefore \frac{1}{1+r} = \frac{5}{6}$; $1+r = \frac{6}{5}$; $r = \frac{1}{5} = 20\%$.

DENOTE i'S SAVING BY s_i:

$z_A = 24 = (1+r)s_A = \frac{6}{5}s_A$; $\therefore s_A = 20$ ← LEND

$s_B = -20$ IN EQUIL'M. ← BORROW

$\therefore x_{A0} = \mathring{x}_{A0} - 20 = 30 - 20 = 10$

$x_{B0} = \mathring{x}_{B0} + 20 = 60 + 20 = 80$

$(x_{A0}, x_{AH}, x_{AL}) = (10, 54, 54)$

$(x_{B0}, x_{BH}, x_{BL}) = (80, 36, 36)$

(d) NOT PARETO: THE ALLOCATION IS NOT AS IN (a).

ALSO $MRS^A_H = \frac{30}{54} > \frac{15}{36} = MRS^B_H$

$MRS^A_L = \frac{15}{54} < \frac{15}{36} = MRS^B_L$.

[PARETO IMPROVEMENT BELOW]

(e) $d_1 = \begin{bmatrix} 1+r \\ 1+r \end{bmatrix}$, $d_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

← p_H, p_L ARE ARROW-DEBREU PRICES;

PRICES $q_1 = (1+r)p_H + (1+r)p_L$

$q_2 = 2p_H + p_L$

r IS ~~COMPLETE-MARKETS~~ EQUIL'M INTEREST RATE (NOT GENERALLY SAME AS r ABOVE)

THE ARROW-DEBREU INTEREST RATE SATISFIES

$\frac{1}{1+r} = p_H + p_L$; i.e., $\frac{1}{1+r} = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$

$\therefore 1+r = \frac{6}{5}$, $r = \frac{1}{5} = 20\%$, SAME AS CREDIT-~~MARKET~~ ONLY INTEREST RATE, BUT THAT'S JUST COINCIDENCE.

$$\therefore \quad q_1 = \left(\tfrac{6}{5}\right)\left(\tfrac{1}{2}\right) + \left(\tfrac{6}{5}\right)\left(\tfrac{1}{3}\right) = \tfrac{3}{5} + \tfrac{2}{5} = 1, \text{ AS IT MUST}$$

$$q_2 = (2)\left(\tfrac{1}{2}\right) + (1)\left(\tfrac{1}{3}\right) = 1 + \tfrac{1}{3} = \tfrac{4}{3}$$

AMY'S CONSUMPTION PLAN IS $(x_{A0}, x_{AH}, x_{AL}) = (10, 60, 45)$,

SO $\overset{\Delta}{x}_{AH} = 30$, $\overset{\Delta}{x}_{AL} = 15$;

$$\therefore \begin{bmatrix} \overset{\Delta}{x}_{AH} \\ \overset{\Delta}{x}_{AL} \end{bmatrix} = \begin{bmatrix} 30 \\ 15 \end{bmatrix} = \begin{bmatrix} \frac{6}{5} \\ \frac{6}{5} \end{bmatrix} y_1^A + \begin{bmatrix} 2 \\ 1 \end{bmatrix} y_2^A$$

THE SOLUTION HERE IS $y_1^A = 0$, $y_2^A = 15$,

$$\therefore x_{A0} = \mathring{x}_{A0} - q_1 y_{A1} - q_2 y_{A2}$$
$$= 30 - (1)(0) - \left(\tfrac{4}{3}\right)(15) = 10.$$

SIMILARLY, $y_{B1} = 0$ AND $y_{B2} = -15$,

$$\therefore x_{B0} = \mathring{x}_{B0} - q_1 y_{B1} - q_2 y_{B2}$$
$$= 60 - (1)(0) - \left(\tfrac{4}{3}\right)(-15) = 80.$$

(d) PARETO IMPROVEMENTS ON THE ALLOCATION

$$\bar{x} = (\bar{x}_A, \bar{x}_B) = ((10, 54, 54), (80, 36, 36)):$$

FIRST NOTE THAT THE A-D ALLOCATION

$$\hat{x} = (\hat{x}_A, \hat{x}_B) = ((10, 60, 45), (80, 30, 45))$$

IS PARETO EFFICIENT. HOWEVER, THAT DOES NOT GUARANTEE THAT IT'S A PARETO IMPROVEMENT ON $\bar{x}$.

NEXT, NOTE THAT AT $\bar{x}$ WE HAVE

$$MRS^A_{HL} = \frac{u_{AH}}{u_{AL}} = \frac{30/54}{15/54} = 2,$$

$$MRS^B_{HL} = \frac{u_{BH}}{u_{BL}} = \frac{15/36}{15/36} = 1,$$

AND THAT AT $\hat{x}$ WE HAVE

$$MRS^A_{HL} = \frac{3}{2} < 2 \text{ AND } MRS^B_{HL} = \frac{3}{2} > 1.$$

CONSEQUENTLY, ANY REALLOCATION FROM $\bar{x}$ THAT SATISFIES $\overset{\Delta}{x}_{AH} > 0$ AND $\overset{\Delta}{x}_{BL} > 0$ AND ← AND $\overset{\Delta}{x}_{A0} = \overset{\Delta}{x}_{B0} = 0$

$$\overset{\Delta}{x}_{AL} = -\frac{3}{2}\overset{\Delta}{x}_{AH} \text{ AND } \overset{\Delta}{x}_{BL} = -\frac{3}{2}\overset{\Delta}{x}_{BH}$$

WILL MAKE BOTH A AND B BETTER OFF, SO LONG AS THE RESULTING ALLOCATION STILL SATISFIES

$$MRS^A_{HL} \geq \frac{3}{2} \text{ AND } MRS^B_{HL} \leq \frac{3}{2}.$$

IN PARTICULAR, ANY MULTIPLE OF $(\overset{\Delta}{x}_{AH}, \overset{\Delta}{x}_{AL}) = (+2, -3)$ AND $(\overset{\Delta}{x}_{BH}, \overset{\Delta}{x}_{BL}) = (-2, +3)$ WILL SATISFY THOSE CONDITIONS, UP TO $(\overset{\Delta}{x}_{AH}, \overset{\Delta}{x}_{AL}) = (+6, -9)$ AND $(\overset{\Delta}{x}_{BH}, \overset{\Delta}{x}_{BL}) = (-6, +9)$, BECAUSE THAT REALLOCATION YIELDS $\hat{x}$, AT WHICH $MRS^A_{HL} = MRS^B_{HL} = \frac{3}{2}$.

(3) (a) max $u_1(x, y_1)$ s.t. $x, y_1, y_2, y_3 \geq 0$

AND $C(x) + y_1 + y_2 + y_3 \leq \mathring{y} \quad : \quad \sigma$

$u_2(x, y_2) \geq c_2 \quad : \quad \lambda_2$

$u_3(x, y_3) \geq c_3 \quad : \quad \lambda_3$

FOC (INTERIOR):

(1) x: $\quad u^1_x = C'(x)\sigma - \lambda_2 u^2_x - \lambda_3 u^3_x$

(2) y_1: $\quad u^1_y = \sigma$

(3) y_2: $\quad 0 = \sigma - \lambda_2 u^2_y$

(4) y_3: $\quad 0 = \sigma - \lambda_3 u^3_y$

(1) $\quad u^1_x + \lambda_2 u^2_x + \lambda_3 u^3_x = (MC)\sigma$.

i.e., $\dfrac{u^1_x}{\sigma} + \dfrac{\lambda_2 u^2_x}{\sigma} + \dfrac{\lambda_3 u^3_x}{\sigma} = MC$

i.e., $\dfrac{u^1_x}{u^1_y} + \dfrac{\lambda_2 u^2_x}{\lambda_2 u^2_y} + \dfrac{\lambda_3 u^3_x}{\lambda_3 u^3_y} = MC$

i.e., $MRS_1 + MRS_2 + MRS_3 = MC$.

(b) $u_1 = y_1 + \ln x \quad u_2 = y_2 + 2\ln x \quad u_3 = y_3 + 3\ln x$

$MC = 3$; $MRS_1 = \frac{1}{x}$, $MRS_2 = \frac{2}{x}$, $MRS_3 = \frac{3}{x}$.

$\Sigma MRS_i = MC$: $\frac{1}{x} + \frac{2}{x} + \frac{3}{x} = 3$; i.e., $\frac{6}{x} = 3$; $\boxed{x = 2}$.

y_1, y_2, y_3 MUST SATISFY $y_1 + y_2 + y_3 = \mathring{y} - 6 = 300 - 6 = 294$, BUT OTHERWISE UNRESTRICTED.

(c) THE LINDAHL ALLOCATION IS PARETO OPTIMAL, $\therefore x = 2$. THE LINDAHL PRICES ARE THE MRS's AT THE LINDAHL ALLOCATION:

$$p_1 = MRS_1 = \frac{1}{2}, \quad p_2 = MRS_2 = 1, \quad p_3 = MRS_3 = \frac{3}{2}.$$

THEREFORE

$$y_1 = \mathring{y}_1 - p_1 x = 100 - 1 = 99$$
$$y_2 = \mathring{y}_2 - p_2 x = 100 - 2 = 98$$
$$y_3 = \mathring{y}_3 - p_3 x = 100 - 3 = 97.$$

(d) $\tilde{u}_i(t_1, t_2, t_3) = \mathring{y}_i - t_i + \alpha_i \ln\left(\frac{1}{3}\right)(t_1 + t_2 + t_3), \quad i = 1,2,3.$

$$\frac{\partial \tilde{u}_i}{\partial t_i} = -1 + \alpha_i \frac{1/3}{\frac{1}{3}(t_1+t_2+t_3)} = -1 + \frac{\alpha_i}{3x}$$

i's FOC IS $\frac{\partial \tilde{u}_i}{\partial t_i} \leqq 0$ AND $\frac{\partial \tilde{u}_i}{\partial t} = 0$ IF $t_i > 0$;

i.e., $-1 + \frac{\alpha_i}{3x} \leqq 0$; i.e., $\frac{\alpha_i}{3x} \leqq 1$; i.e., $\frac{\alpha_i}{x} \leqq 3$

i.e., $MRS_i \leqq MC$ FOR EACH i.

EQUIVALENTLY, $x \geqq \frac{\alpha_i}{3}$ FOR EACH i, AND $t_i > 0 \Rightarrow x = \frac{\alpha_i}{3}$.

THUS, $x \geqq \max\left\{\frac{\alpha_1}{3}, \frac{\alpha_2}{3}, \frac{\alpha_3}{3}\right\}$

$= \max\left\{\frac{1}{3}, \frac{2}{3}, 1\right\} = 1$, AND $t_1 = t_2 = 0$.

IN ORDER TO HAVE $x > 0$ WE NEED $t_i > 0$ FOR SOME i;

$\therefore t_3 > 0$ AND $x = \frac{\alpha_3}{3} = 1$.

SUMMARIZING: $x = 1$; $t_1 = 0$, $t_2 = 0$, $t_3 = 3$.

(e) $x = r_1 + r_2 + r_3$

$t_1 = (1 + r_2 - r_3)x \qquad t_2 = (1 + r_3 - r_1)x \qquad t_3 = (1 + r_1 - r_2)x$

$$\tilde{u}_1(r_1, r_2, r_3) = \mathring{y}_i - (1 + r_2 - r_3)(r_1 + r_2 + r_3) + \alpha_1 \ln(r_1 + r_2 + r_3)$$

$$\frac{\partial \tilde{u}_1}{\partial r_1} = -(1 + r_2 - r_3) + \frac{\alpha_1}{r_1 + r_2 + r_3} = -(1 + r_2 - r_3) + \frac{\alpha_1}{x}$$

FOC: $\frac{\alpha_1}{x} = 1 + r_2 - r_3$; SIMILARLY FOR r_2 AND r_3.

THEREFORE THE THREE PLAYERS' FOC'S ARE:

$$\left.\begin{array}{ll} (1) & \frac{\alpha_1}{x} = 1 + r_2 - r_3 \\ (2) & \frac{\alpha_2}{x} = 1 + r_3 - r_1 \\ (3) & \frac{\alpha_3}{x} = 1 + r_1 - r_2 \end{array}\right\}$$

ADDING THESE (ASSUMING ALL THREE ARE TRUE):

$$\frac{\alpha_1}{x} + \frac{\alpha_2}{x} + \frac{\alpha_3}{x} = 3$$

i.e., $x = \frac{1}{3}(\alpha_1 + \alpha_2 + \alpha_3) = \frac{6}{3} = 2$.

NOW WE WANT TO SOLVE FOR r_1, r_2, r_3:

(1) $\frac{1}{2} = 1 + r_2 - r_3$; i.e., $r_3 = r_2 + \frac{1}{2}$.

(2) $1 = 1 + r_3 - r_1$; i.e., $r_1 = r_3 = r_2 + \frac{1}{2}$.

WE ALREADY KNOW THAT $r_1 + r_2 + r_3 = x = 2$,

BUT NOW WE ALSO HAVE

$$r_1 + r_2 + r_3 = (r_2 + \tfrac{1}{2}) + r_2 + (r_2 + \tfrac{1}{2})$$
$$= 3r_2 + 1.$$

$\therefore 3r_2 + 1 = 2$; i.e., $3r_2 = 1$; i.e., $r_2 = \frac{1}{3}$.

$\therefore r_1 = \frac{5}{6},\ r_2 = \frac{1}{3},\ r_3 = \frac{5}{6}$; $r_1 + r_2 + r_3 = x = 2$.

$\therefore t_1 = \frac{1}{2}x,\ t_2 = x,\ t_3 = \frac{3}{2}x$,

WHICH ARE THE LINDAHL PAYMENTS AT $x = 2$,

THE LINDAHL LEVEL OF x, AS IN (c):

$t_1 = 1,\ t_2 = 2,\ t_3 = 3$.