ECON 501B MIDTERM EXAM SOLUTIONS

(1) $u_A(x,y) = x^3 y$ $\quad MRS_A = 3\frac{y_A}{x_A}$

$u_B(x,y) = x + 2y$ $\quad MRS_B = \frac{1}{2}$

$(\overset{\circ}{x}, \overset{\circ}{y}) = (30, 20)$

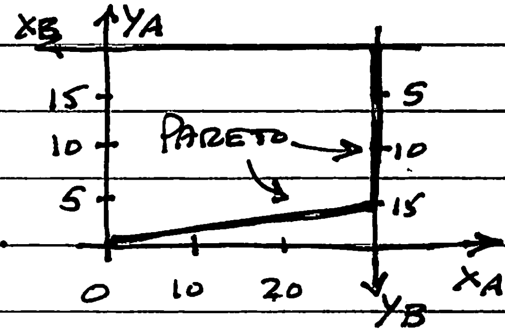
(a) $MRS_A = MRS_B$: $3\frac{y_A}{x_A} = \frac{1}{2}$; i.e., $y_A = \frac{1}{6}x_A$, IF INTERIOR.

IF $y_A = 0$ OR $x_B = 0$: $MRS_A \geqq MRS_B$

i.e., $y_A \geqq \frac{1}{6}x_A$; $\therefore y_A > 0$, $x_B = 0$.

IF $x_A = 0$ OR $y_B = 0$: $MRS_A \leqq MRS_B$:

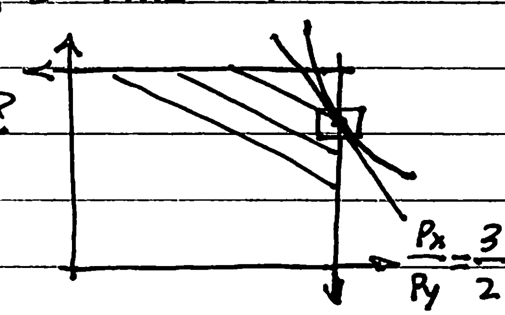
i.e., $y_A \leqq \frac{1}{6}x_A$ — NO SUCH ALLOCATION.



(b) THE INITIAL ALLOCATION IS PARETO OPTIMAL, THEREFORE IT IS AN EQUILIBRIUM ALLOCATION. AT THIS ALLOCATION WE HAVE $MRS_A = \frac{3}{2}$ AND $MRS_B = \frac{1}{2}$. IN ORDER FOR A TO CHOOSE $(30, 15)$ WE MUST HAVE $\frac{P_x}{P_y} = \frac{3}{2}$, AND AT THESE PRICES B WILL CHOOSE $(0, 5)$, SO THIS IS AN EQUILIBRIUM.

ARE THERE ANY OTHER EQUILIBRIA?

IF $\frac{P_x}{P_y} \neq \frac{3}{2}$, A WILL CHOOSE A DIFFERENT BUNDLE. IF $\frac{P_x}{P_y} > \frac{1}{2}$, B WILL STILL CHOOSE $(0, 5)$, SO MARKETS WILL CLEAR ONLY IF A CHOOSES $(x_A, y_A) = (30, 15)$, WHICH WILL HAPPEN ONLY IF $\frac{P_x}{P_y} = \frac{3}{2}$. IF $\frac{P_x}{P_y} \leqq \frac{1}{2}$, A WILL CHOOSE $x_A > \overset{\circ}{x}_A$ AND B WILL CHOOSE $x_B \geqq \overset{\circ}{x}_B$, SO MARKETS WILL NOT CLEAR.



YOU COULD, ALTERNATIVELY, SHOW THAT THIS EQUILIBRIUM IS UNIQUE BY INVOKING THE FIRST WELFARE THEOREM: ANY OTHER EQUILIBRIUM ALLOCATION IS PARETO OPTIMAL; BUT IT MUST ALSO LIE "NW" OF THE INITIAL ALLOCATION (B SELLS PEANUTS, BUYS BEER), AND THERE ARE NO SUCH PARETO ALLOCATIONS.

(2) $u_A(x, y_A) = y_A + 6x - \frac{1}{2}x^2$ $\quad MRS_A = 6 - x$

$u_B(x, y_B) = y_B - x^2$ $\quad MRS_B = -2x$

(a) $\max_{x, y_A, y_B} u_A(x, y_A) = y_A + 6x - \frac{1}{2}x^2$ s.t. $x, y_A, y_B \geqq 0$

AND TO $y_A + y_B \leqq \mathring{y}$ $\quad \sigma \leftarrow$ LAGRANGE

$u_B(x, y_B) \geqq \bar{u}_B$ $\quad \lambda \leftarrow$ MULTIPLIERS

FOMC: (INTERIOR)

x: $u_{Ax} = -\lambda\underbrace{(-2x)}_{u_{Bx}} = 2\lambda x$ $\quad 6 - x = 2\lambda x$

y_A: $u_{Ay} = \sigma$ $\quad 1 = \sigma$

y_B: $0 = \sigma - \lambda u_{By}$ $\quad \sigma = \lambda$

$\lambda = \sigma = 1$

$6 - x = 2x$

i.e., $3x = 6$

$\boxed{x = 2}$

PARETO ALLOCATIONS:

THE (x, y_A, y_B) THAT SATISFY $x = 2$ AND $y_A + y_B = \mathring{y}$.

(b) AT $x = 2$: $MRS_A = 4$ AND $MRS_B = -4$.

IF $x > 2$, THEN $MRS_A < 4$ AND $MRS_B < -4$ $(|MRS_B| > 4)$. THIS TELLS US THAT BART WOULD BE WILLING TO PAY MORE THAN \$4 FOR A ONE-UNIT REDUCTION OF X AND ARNIE WOULD ACCEPT LESS THAN \$4 FOR A ONE-UNIT REDUCTION. SO WE COULD REDUCE x (i.e., $\Delta x < 0$), KEEPING $x + \Delta x \geqq 2$, AND TRANSFER $(\$4)(-\Delta x)$ FROM BART TO ARNIE, AND THEY WILL BOTH BE BETTER OFF.

SIMILARLY, IF $x < 2$ THEN $MRS_A > 4$ AND $MRS_B > -4$ $(|MRS_B| < 4)$. THIS TELLS US THAT ARNIE WOULD BE WILLING TO PAY MORE THAN \$4 FOR A UNIT INCREASE IN X AND BART WOULD ACCEPT LESS THAN \$4. SO WE CAN INCREASE x (i.e., $\Delta x > 0$), KEEPING $x + \Delta x \leqq 2$, AND TRANSFER $(\$4)(\Delta x)$ FROM ARNIE TO BART, AND THEY WILL BOTH BE BETTER OFF.

(3) $u_A(x,y) = x^2 y$ $\quad MRS_A = 2\frac{y_A}{x_A}$

$u_B(x,y) = xy^2$ $\quad MRS_B = (\frac{1}{2})\frac{y_B}{x_B}$ $\quad (\mathring{x}, \mathring{y}) = (66, 12)$

$q = f(z) = 12\sqrt{z}$ $\quad f'(z) = \frac{6}{\sqrt{z}}$

(a) $x_A = 40$, $x_B = 10$; $\therefore$ $z = \mathring{x} - (x_A + x_B) = 66 - 50 = 16$.

$\therefore$ $q = f(16) = 12\sqrt{16} = 48$ AND $f'(z) = \frac{6}{4} = \frac{3}{2}$.

PARETO OPTIMALITY REQUIRES THAT

$MRS_A = f'(z) = \frac{3}{2}$; i.e., $2\frac{y_A}{x_A} = \frac{3}{2}$; $y_A = \frac{3}{4}x_A = 30$.

$MRS_B = f'(z) = \frac{3}{2}$; i.e., $(\frac{1}{2})\frac{y_B}{x_B} = \frac{3}{2}$; $y_B = 3x_B = 30$.

IT ALSO REQUIRES THAT $y_A + y_B = \mathring{y} + q = 12 + 48 = 60$, WHICH IS CONSISTENT WITH THE ABOVE.

$\therefore$ THE ALLOCATION $((x_A, y_A), (x_B, y_B), z, q) = ((40,30), (10,30), 16, 48)$ IS PARETO OPTIMAL

(b) NOW $(\mathring{x}_A, \mathring{y}_A) = (44, 0)$ AND $(\mathring{x}_B, \mathring{y}_B) = (22, 12)$ AND ANN RECEIVES ALL PROFIT. LET $p_x = \$3$ AND $p_y = \$2$.

THE FIRM MAXIMIZES PROFIT AT A z THAT SATISFIES $f'(z) = \frac{p_x}{p_y}$ — i.e., AT $z = 16$ AND $q = 48$, AS IN (a).

THE RESULTING PROFIT IS $\pi = p_y q - p_x z = (2)(48) - (3)(16) = 48$.

ANN CHOOSES (x_A, y_A) TO SATISFY $MRS_A = \frac{p_x}{p_y}$ — i.e., $2\frac{y_A}{x_A} = \frac{3}{2}$ — $\quad y_A = \frac{3}{4}x_A$

AND $p_x x_A + p_y y_A = p_x \mathring{x}_A + p_y \mathring{y}_A + \pi$;

i.e., $3x_A + 2y_A = (3)(44) + (2)(0) + 48 = 180$

COMBINING, WE HAVE $3x_A + \frac{3}{2}x_A = 180$; i.e., $\frac{9}{2}x_A = 180$; $x_A = 40$, $y_A = 30$.

ANN SELLS 4 UNITS OF X TO HER FIRM AND BUYS 30 UNITS OF Y (6 FROM SALES OF X, 24 FROM PROFIT).

BOB CHOOSES (x_B, y_B) TO SATISFY $MRS_B = \frac{p_x}{p_y}$ — i.e., $(\frac{1}{2})\frac{y_B}{x_B} = \frac{3}{2}$ — $\quad y_B = 3x_B$

AND $p_x x_B + p_y y_B = p_x \mathring{x}_B + p_y \mathring{y}_B$,

i.e., $3x_B + 2y_B = (3)(22) + (2)(12) = 66 + 24 = 90$.

COMBINING: $3x_B + 6x_B = 90$; i.e., $x_B = 10$, $y_B = 30$.

BOB SELLS 12 UNITS OF X AND BUYS 18 UNITS OF Y

(c) ANN DOES NOT CHOOSE THE PRODUCTION PLAN (z, q) SO AS TO MAXIMIZE HER UTILITY, BUT INSTEAD SO AS TO MAXIMIZE PROFIT. HOWEVER, FISHER'S SEPARATION THEOREM TELLS US THAT THIS WILL ACTUALLY BE THE RIGHT CHOICE TO MAXIMIZE HER UTILITY. SHE (AND WE) CAN SEPARATE HER PRODUCTION AND CONSUMPTION DECISIONS.

(d) LET $(x_A, y_A) = (20, 28)$ AND $(x_B, y_B) = (10, 56)$. THEN

$$MRS_A = 2\left(\frac{28}{20}\right) = 2.8 \text{ AND } MRS_B = \left(\frac{1}{2}\right)\left(\frac{56}{10}\right) = 2.8,$$

AND WE ALSO HAVE

$$z = \mathring{x} - (x_A + x_B) = 66 - 30 = 36 \text{ AND}$$

$$q = f(z) = 12\sqrt{36} = 72, \text{ AND } f'(z) = \frac{6}{\sqrt{36}} = 1.$$

WE THEREFORE HAVE $\mathring{y} + q = 12 + 72 = 84 = y_A + y_B$, SO THE CONSUMPTION PLAN IS FEASIBLE: THIS FEASIBLE PRODUCTION PLAN $(z, q) = (36, 72)$ SUPPORTS IT.

BUT THIS PLAN IS <u>NOT</u> PARETO OPTIMAL: $MRS_A = MRS_B \neq f'(z)$. SINCE $f'(z) < MRS_A, MRS_B$, WE CAN CREATE A PARETO IMPROVEMENT BY DECREASING z AND q, INCREASING x_A AND x_B, AND DECREASING y_A AND y_B. BUT NOTE THAT MOVING TO THE PARETO ALLOCATION IN (a) IS <u>NOT</u> A PARETO IMPROVEMENT: BOB IS MADE MUCH WORSE OFF.

A PARETO IMPROVEMENT: $z = 25$, $q = 60$ $(\Delta z = -11, \Delta q = -12)$

$(x_A, y_A) = (25, 22)$, $u_A = (625)(22) = 13{,}750 > u_A(20, 28)$

$(x_B, y_B) = (16, 50)$, $u_B = (16)(2500) = 40{,}000 > u_B(10, 56)$.

THIS IS FEASIBLE: $x_A + x_B + z = 66$ AND $y_A + y_B = 72 = \mathring{y} + q$.