ECON 501B FINAL EXAM SOLUTIONS
FALL 2012

① (a) INTERIOR PARETO: $\sum_{i=1}^{n} MRS^i = MC$ & $\sum_{i=1}^{n} y_i + C(x) = \mathring{y}$.

THIS YIELDS $30 - 3x = 6$; i.e., $x = 8$

AND ∴ $y_A + y_B + y_C = \sum \mathring{y}_i - 48$.

(b) LET x_i DENOTE i's PURCHASE.

A WILL CHOOSE x_A TO SATISFY $MRS_A = MC$ (OR P), i.e., $15 - x = 6$

i.e., $15 - (x_A + x_B + x_C) = 6$, UNLESS THIS YIELDS $x_A < 0$, IN WHICH CASE $x_A = 0$. SIMILARLY FOR B AND C.

THUS, THE REACTION FUNCTIONS ARE

$x_A = \max\{0, 9 - x_B - x_C\}$, $x_B = \max\{0, 4 - x_A - x_C\}$, $x_C = 0$.

NASH EQUILIBRIUM: $x_C = 0$;

IF $x_B > 0$, THEN $x_A + x_B = 4$ AND $x_A + x_B = 9$, WHICH IS NOT POSSIBLE; ∴ $x_B = 0$.

SINCE $x_B = 0$ AND $x_C = 0$, WE HAVE $x_A = 9$, SO $x = 9$. LARGER THAN PARETO!

(c) PARETO OPTIMALITY REQUIRES $x = 8$, A DECREASE OF ONE UNIT. THIS WILL YIELD AN INCREASE OF 6 y-UNITS RELEASED FROM PRODUCING THE x-GOOD. HOW SHOULD WE DISTRIBUTE THOSE 6 UNITS? LET'S SAY ANN'S PAYMENT FOR THE x-GOOD IS REDUCED BY THOSE 6 y-UNITS.

WE STILL NEED ADDITIONAL TRANSFERS, BECAUSE BOTH u_A AND u_B HAVE BEEN REDUCED BY REDUCING x.

WE HAVE, <u>AT $x = 8$</u>: $MRS^A = 7$, $MRS^B = 2$, $MRS^C = -3$.

∴ EACH PERSON'S UTILITY WILL BE INCREASED IF $\Delta x = -1$ AND $\Delta y_A = 7$, $\Delta y_B = 2$, $\Delta y_C = -3$, WHICH WE CAN ACHIEVE WITH $t_A = \$1$ (SO $\Delta y_A = \$7$), $t_B = \$2$, AND $t_C = -\$3$.

(d) Now our Pareto marginal condition $\sum MRS^i = MC$ is $60 - 6x = 12$ — i.e., $x = 8$, just as before.

But now the cost of the spray, \$12, exceeds MRS_B as well as MRS_C, so we again have $x_B = 0$ for both B's and $x_C = 0$ for both C's. For each A, the reaction function is

$$x_A = \max\{0, 15 - 12 - x_{A'} - x_B - x_{B'} - x_C - x_{C'}\}$$

$$= \max\{0, 3 - x_{A'}\}, \text{ given that } x_B = x_{B'} = x_C = x_{C'} = 0.$$

Therefore we have $x_A + x_{A'} = 3$ — i.e., the two A's will purchase a total of $x = 3$, and therefore $x = 3$ is the total amount of spray. Note that then $MRS_A = MRS_{A'} = 12 = MC$.

With 3 people we had $x = 9$ (while Pareto was $x = 8$), and with 3 identical people added (i.e., replicating the 3-person economy) we have $x = 3$ while Pareto is still $x = 8$.

② $u^A(x_0, x_H, x_L) = x_0 + x_H - \frac{1}{40}x_H^2 + x_L - \frac{1}{18}x_L^2 \qquad \mathring{x}^A = (12, 18, 0)$

$u^B(x_0, x_H, x_L) = x_0 + x_H - \frac{1}{32}x_H^2 + x_L - \frac{1}{30}x_L^2 \qquad \mathring{x}^B = (12, 0, 16)$

$MRS_H^A = 1 - \frac{1}{20}x_H^A \qquad MRS_L^A = 1 - \frac{1}{9}x_L^A$

$MRS_H^B = 1 - \frac{1}{16}x_H^B \qquad MRS_L^B = 1 - \frac{1}{15}x_L^B$

(a) PARETO:

$MRS_H^A = MRS_H^B \qquad\qquad MRS_L^A = MRS_L^B$

$1 - \frac{1}{20}x_H^A = 1 - \frac{1}{16}x_H^B \qquad\qquad 1 - \frac{1}{9}x_L^A = 1 - \frac{1}{15}x_L^B$

$\frac{1}{20}x_H^A = \frac{1}{16}x_H^B \qquad\qquad \frac{1}{9}x_L^A = \frac{1}{15}x_L^B$

$x_H^B = \frac{4}{5}x_H^A \qquad\qquad x_L^A = \frac{3}{5}x_L^B$

$x_H^A + x_H^B = 18 \qquad\qquad x_L^A + x_L^B = 16$

$\boxed{\therefore x_H^A = 10,\ x_H^B = 8 \qquad \therefore x_L^A = 6,\ x_L^B = 10 \qquad \text{AND } x_0^A + x_0^B = 24}$

(b) LET $p_0 = 1$; THEN $\left.\begin{array}{l} p_H = MRS_H^i,\ i = A, B \\ p_L = MRS_L^i,\ i = A, B \end{array}\right\}$ AT THE PARETO ALLOCATIONS

i.e., $\boxed{p_H = \frac{1}{2},\ p_L = \frac{1}{3}}$ $\boxed{\begin{array}{l} x_0^A = 12 - (\frac{1}{2})(-8) - (\frac{1}{3})(6) = 14 \\ x_0^B = 12 - (\frac{1}{2})(8) - (\frac{1}{3})(-6) = 10 \end{array}}$

(c) ALLOCATIONS IN THE CREDIT MARKET MUST SATISFY $x_H^i - \mathring{x}_H^i = x_L^i - \mathring{x}_L^i$, i.e., CONSUMERS' TRADES WILL BE THE SAME IN EACH STATE. BUT WE FOUND IN (a) THAT PARETO EFFICIENCY REQUIRES TRADES THAT DIFFER BY STATE: $x_H^A - \mathring{x}_H^A = -8$, $x_L^A - \mathring{x}_L^A = 6$, AND THE OPPOSITE FOR B.

IN FACT, THE CREDIT MARKET WILL BE <u>COMPLETELY</u> INEFFECTIVE, BECAUSE EACH PERSON HAS A STATE s IN WHICH $\mathring{x}_s^i = 0$, SO HE WOULD BE UNABLE TO REPAY A LOAN. SO THERE IS NO TRADE.

(d) $d_1 = \begin{bmatrix}1\\1\end{bmatrix}$, $d_2 = \begin{bmatrix}1\\0\end{bmatrix}$

$$p_1 = p_H + p_L = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}; \quad \frac{1}{1+r} = p_1 = \frac{5}{6}, \quad \therefore r = \frac{1}{5} = 20\%$$

$$p_2 = p_H = \frac{1}{2}.$$

THE EQUILIBRIUM ALLOCATION IS THE ARROW-DEBREU ALLOCATION IN (b). DENOTING PERSON i'S PURCHASE OF SECURITY k BY y_k^i, WE HAVE

$$\begin{bmatrix} x_H^i - \overset{\circ}{x}{}_H^i \\ x_L^i - \overset{\circ}{x}{}_L^i \end{bmatrix} = y_1^i \begin{bmatrix}1\\1\end{bmatrix} + y_2^i \begin{bmatrix}1\\0\end{bmatrix}.$$

FOR A:

$$\begin{bmatrix} x_H^A - \overset{\circ}{x}{}_H^A \\ x_L^A - \overset{\circ}{x}{}_L^A \end{bmatrix} = \begin{bmatrix}-8\\6\end{bmatrix}, \quad \therefore y_1^A = 6 \text{ AND } y_2^A = -14.$$

THIS COSTS A $p_1 y_1^A = \left(\frac{5}{6}\right)6 = \$5$ SO A IS SAVING \$5 TODAY.

FOR B:

$$\begin{bmatrix} x_H^B - \overset{\circ}{x}{}_H^B \\ x_L^B - \overset{\circ}{x}{}_L^B \end{bmatrix} = \begin{bmatrix}8\\-6\end{bmatrix}, \quad \therefore y_1^B = -6 \text{ AND } y_2^B = 14.$$

B IS BORROWING \$5 TODAY, REPAYING \$6 TOMORROW.

ALTERNATIVELY, DEFINE d_1 BY $d_1 = \begin{bmatrix}1+r\\1+r\end{bmatrix}$ WITH $p_1 \equiv 1$. WE AGAIN GET $r = \frac{1}{5}$, BUT NOW WE GET

$y_1^A = 5$ (A'S SAVING TODAY) AND $y_2^A = -14$

$y_1^B = -5$ (B'S BORROWING TODAY) AND $y_2^B = 14$.

NOTE THAT IN THIS CASE WE HAVE

$$p_1 = (1+r)p_H + (1+r)p_L = \frac{6}{5}(p_H + p_L) = \frac{6}{5}\left(\frac{1}{2} + \frac{1}{3}\right) = 1.$$

VIA ARROW'S PRICING FORMULA

③ $u(x,y) = y + 8x - \frac{1}{2}x^2 \quad MRS = 8 - x \quad \therefore\ x(p) = 8 - p.$

$q = f(z) = 10\sqrt{z} \quad f'(z) = \frac{5}{\sqrt{z}} \quad MC = \frac{1}{f'(z)} = \frac{1}{5}\sqrt{z} = \frac{1}{50}q.$

(a) SUPPOSE $z' \neq z''$ ARE THE INPUT LEVELS OF TWO FIRMS. LET $q' = f(z')$ AND $q'' = f(z'')$, AND LET $\bar{z} = \frac{1}{2}(z' + z'')$ AND $\bar{q} = f(\bar{z})$. SINCE f IS STRICTLY CONCAVE, WE HAVE

$$\bar{q} = f(\bar{z}) = f\left(\tfrac{1}{2}z' + \tfrac{1}{2}z''\right) > \tfrac{1}{2}f(z') + \tfrac{1}{2}f(z'') = \tfrac{1}{2}q' + \tfrac{1}{2}q''.$$

IF EACH OF THE TWO FIRMS USES $\bar{z}$ TO PRODUCE $\bar{q}$, THEN TOGETHER THEY USE $2\bar{z} = z' + z''$ AND PRODUCE $2\bar{q} > q' + q''$. SO THE SAME AMOUNT OF INPUT AS BEFORE IS USED TO PRODUCE MORE WIDGETS, AND THE ADDITIONAL WIDGETS CAN BE AMONG ONE OR MORE CONSUMERS WITHOUT ANYONE'S CONSUMPTION BEING REDUCED — A PARETO IMPROVEMENT.

(b) EACH CONSUMER'S DEMAND FOR WIDGETS IS $x(p) = 8 - p$, AS ABOVE. (THE INCOME EFFECT ON WIDGET DEMAND IS ZERO, SO THE PROFITS EARNED BY THE CONSUMERS WHO OWN FIRMS DOES NOT AFFECT THEIR DEMAND FOR WIDGETS.)

$\therefore$ MARKET DEMAND IS $X(p) = 1000\,x(p) = 8000 - 1000p.$

THE FIRMS' SUPPLY FUNCTIONS FOR WIDGETS SATISFY $MC = p$. EACH FIRM'S MC FUNCTION IS $MC = \frac{1}{50}q$, AS ABOVE; SO THE FIRM'S SUPPLY FUNCTION IS $q = 50p$.

$\therefore$ MARKET SUPPLY IS $Q(p) = 20\,q(p) = 1000p.$

EQUILIBRIUM (DEMAND = SUPPLY IN BOTH MARKETS):

WIDGETS: $8000 - 1000p = 1000p \quad \therefore \boxed{p = 4}, \quad Q = 4000$

SIMOLEANS: MUST CLEAR, BY WALRAS' LAW, BUT CHECK BELOW.

EACH FIRM: $q = 50p = 200, \quad z = \frac{1}{100} q^2 = 400$

$$\pi = (200)(4) - 400 = 400$$

EACH CONSUMER: $x = 8 - p = 4$

$$y = \begin{cases} 100 - (4)(4) = 84 & \text{IF NOT A FIRM OWNER} \\ 84 + 400 = 484 & \text{IF A FIRM OWNER} \end{cases}$$

CHECK FOR MARKET-CLEARING:

$X = 1000x = 4000 = Q$ WIDGET MARKET CLEARS

$Y = (1000)(84) + (20)(400) = 92{,}000$

$Z = 20z = (20)(400) = 8{,}000$

$Y + Z = 100{,}000 = \mathring{y}$ SIMOLEAN MARKET CLEARS

(c) EACH CONSUMER'S $MRS = 4$.
EACH FIRM'S $MRTS = \frac{1}{g'(z)} = \frac{1}{50} q = \frac{200}{50} = 4$. } MARGINAL CONDITIONS FOR PARETO ARE SATISFIED

WE SHOWED ABOVE THAT $X = Q$ AND $Y + Z = \mathring{y}$. } RESOURCES EXACTLY USED: C.S. CONDITIONS SATISFIED

$\therefore$ THE EQUILIBRIUM IS PARETO EFFICIENT.

(4) $(\mathring{x}_J, \mathring{y}_J) = (\mathring{x}_E, \mathring{y}_E) = (12, 0)$ AND $(\mathring{x}_K, \mathring{y}_K) = (0, 12)$.

$u(x,y) = xy$ FOR EVERYONE.

LET (x_S, y_S) DENOTE $\sum_{i \in S}(x_i, y_i)$ FOR ANY COALITION S AND ANY ALLOCATION TO S.

WE HAVE $(\mathring{x}_S, \mathring{y}_S) = (24, 12)$ FOR $S = \{J, E, K\}$.

(a) PARETO EFFIENCY FOR A COALITION S REQUIRES THAT MRS^i IS THE SAME FOR ALL $i \in S$, i.e., FOR SOME r_S WE MUST HAVE $\frac{y_i}{x_i} = r_S$ — i.e., $y_i = r_S x_i$ — FOR ALL $i \in S$.

BUT THEN WE HAVE

$$\sum_{i \in S} y_i = r_S \sum_{i \in S} x_i \text{ — i.e., } y_S = r_S x_S.$$

WE ALSO MUST HAVE $x_S = \mathring{x}_S$ AND $y_S = \mathring{y}_S$; $\therefore r_S = \frac{\mathring{y}_S}{\mathring{x}_S}$.

UTILITY FRONTIERS:

FOR EACH $i \in S$ WE HAVE $u_i = x_i y_i = x_i(r_S x_i) = r_S x_i^2$,

AND $\therefore \sqrt{u_i} = \sqrt{r_S}\, x_i$.

$$\therefore \sum_{i \in S} \sqrt{u_i} = \sum_{i \in S} \sqrt{r_S}\, x_i = \sqrt{r_S} \sum_{i \in S} x_i = \sqrt{r_S}\, \mathring{x}_S = \sqrt{\mathring{x}_S \mathring{y}_S}$$

IS THE UTILITY FRONTIER FOR COALITION S.

FOR COALITIONS OF SIZE $n=1$:

$u_J = 0$, $u_E = 0$, $u_K = 0$ (EACH HAS 0 OF ONE GOOD.)

FOR COALITIONS OF SIZE $n=2$:

$S = \{J, E\}$: $\mathring{y}_S = 0$, SO $\mathring{x}_S \mathring{y}_S = 0$, SO $\sqrt{u_J} + \sqrt{u_E} = 0$.

$S = \{J, K\}$: $(\mathring{x}_S, \mathring{y}_S) = (12, 12)$, $\sqrt{u_J} + \sqrt{u_K} = \sqrt{144} = 12$.

$S = \{E, K\}$: SAME AS $\{J, K\}$, $\sqrt{u_E} + \sqrt{u_K} = \sqrt{144} = 12$.

FOR $n=3$;

$S = \{J, E, K\}$: $(\mathring{x}_S, \mathring{y}_S) = (24, 12)$, SO $\sum \sqrt{u_i} = \sqrt{(24)(12)} = 12\sqrt{2}$.

(b) THE PROPOSED ALLOCATION HAS

$$\sqrt{u_J} + \sqrt{u_K} = \sqrt{18} + \sqrt{(16)(8)} = 3\sqrt{2} + 8\sqrt{2} = 11\sqrt{2} > 12,$$

SO $\{J, K\}$ CAN'T IMPROVE ON THE PROPOSAL;

AND $\sqrt{u_E} + \sqrt{u_K} = \sqrt{2} + 8\sqrt{2} = 9\sqrt{2} > 12,$

SO $\{E, K\}$ CAN'T IMPROVE ON THE PROPOSAL.

CLEARLY $\{J, E\}$, AND THE ONE-PERSON COALITIONS CAN'T IMPROVE.

FINALLY, THE PROPOSAL IS PARETO EFFICIENT FOR $S = \{J, E, K\}$, SINCE $MRS_i = \frac{1}{2}$ FOR ALL i AND ALL RESOURCES ARE USED. ALTERNATIVELY,

$$\sum \sqrt{u_i} = 3\sqrt{2} + \sqrt{2} + 8\sqrt{2} = 12\sqrt{2},$$

SO THE UTILITY PROFILE (u_J, u_E, u_K) IS ON THE UTILITY FRONTIER.

SINCE NO COALITION CAN IMPROVE ON THE PROPOSAL, IT IS IN THE CORE.

(c) THE THEOREM SAYS THAT IF THERE ARE EQUAL NUMBERS OF EACH TYPE OF CONSUMER (AS WELL AS OTHER ASSUMPTIONS), THEN A CORE ALLOCATION WILL GIVE EACH CONSUMER OF A GIVEN TYPE THE SAME BUNDLE. IN THIS "ECONOMY" (AND IN EVERY OTHER WAY!) THERE IS NO ONE ELSE LIKE KRAMER. SO IN ORDER TO APPLY THE THEOREM THERE CAN ONLY BE <u>ONE</u> PERSON OF EACH TYPE — JERRY AND ELAINE MUST BE TREATED AS DISTINCT TYPES, SO THERE ARE THREE TYPES. BUT WITH ONLY ONE PERSON OF EACH TYPE, THE THEOREM'S CONCLUSION IS USELESS.

(d) Core allocations have to be Pareto optimal, so they have to satisfy

$x_i = 2y_i$, $i = J, E, K$ $\therefore u_i = \sqrt{2y_i^2}$ and $y_J + y_E + y_K = 12$.

The coalitions {J}, {E}, {K}, and {J, E} can't improve on <u>any</u> allocation, so they place no restriction on the core allocations. This leaves only the coalitions {J, K} and {E, K}:

$$\sqrt{u_J} + \sqrt{u_K} \geqq 12 \quad \text{and} \quad \sqrt{u_E} + \sqrt{u_K} \geqq 12$$

i.e., $\sqrt{2y_J^2} + \sqrt{2y_K^2} \geqq 12$ and $\sqrt{2y_E^2} + \sqrt{2y_K^2} \geqq 12$

i.e., $\sqrt{2}\,(y_J + y_K) \geqq 12$ and $\sqrt{2}\,(y_E + y_K) \geqq 12$

i.e., $y_J + y_K \geqq \frac{12}{\sqrt{2}} = 6\sqrt{2}$ and $y_E + y_K \geqq \frac{12}{\sqrt{2}} = 6\sqrt{2}$. (*)

Therefore the core allocations are the allocations that satisfy the two inequalities in (*) and $x_i = 2y_i$, $\forall i$, and $y_J + y_E + y_K = 12$.