ECON 501B FALL 2013
MIDTERM EXAM SOLUTIONS

(1) (a) WE HAVE $\tilde{u}_1 = \tilde{u}_3 = \frac{1}{32}$ AND $\tilde{u}_4 = 0$, AND $(\overset{\circ}{x}, \overset{\circ}{y})_{S'} = (1,2)$. PARETO-FOR-$S'$ ALLOCATIONS SATISFY $y_i = 2x_i$ FOR ALL $i \in S'$. AN ALLOCATION TO S' THAT MAKES ALL $i \in S'$ BETTER OFF THAN AT $(\tilde{x}^i, \tilde{y}^i)_N$ IS $(\bar{x}^i, \bar{y}^i) = (\frac{1}{3}, \frac{2}{3})$ FOR $i = 1, 3, 4$. THIS YIELDS $\bar{u}_i = \frac{2}{9} > \hat{u}_i$ FOR EACH $i \in S'$.

(b) LET $(\bar{\bar{x}}^i, \bar{\bar{y}}^i)_N$ BE THE WEA IN WHICH $(\bar{\bar{x}}^i, \bar{\bar{y}}^i) = (\frac{1}{2}, \frac{1}{2})$ FOR ALL $i \in N$. THEN $\bar{\bar{u}}^i = \frac{1}{4}$ FOR ALL $i \in N$, AND WE HAVE

(1) $\bar{\bar{u}}_i = \frac{1}{4} > \frac{1}{32} = \tilde{u}^i$ FOR $i = 1, 3, 4$.

(2) $\bar{\bar{u}}_i = \frac{1}{4} > 0 = \tilde{u}^i$ FOR $i = 2$.

(3) $(\bar{\bar{x}}^i, \bar{\bar{y}}^i)_N$ IS IN THE CORE (BECAUSE EVERY WEA IS IN THE CORE), $\therefore$ NO COALITION CAN IMPROVE ON IT.

(2) $(\mathring{x}, \mathring{y}) = (16, 0)$ AND $u(x,y) = xy$ FOR EACH PERSON.

$q_1 = f_1(z_1) = 2\sqrt{z_1}$ AND $q_2 = f_2(z_2) = \frac{1}{2} z_2$.

WE HAVE $MRS_A = \frac{y_A}{x_A}$ AND $MRS_B = \frac{y_B}{x_B}$,

AND $f_1'(z_1) = \frac{1}{\sqrt{z_1}}$ AND $f_2'(z_2) = \frac{1}{2}$.

(a) PRODUCTION EFFICIENCY REQUIRES (IF $z_2 > 0$):

$f_1'(z_1) = f_2'(z_2)$ — i.e., $\sqrt{z_1} = 2$, SO $z_1 = 4$, $q_1 = 4$.

FEASIBILITY (AND NO WASTE) REQUIRES

(1) $x_A + x_B + z_1 + z_2 = 16$

(2) $y_A + y_B = q_1 + q_2$ (SINCE $\mathring{y} = 0$).

SINCE $z_1 = q_1 = 4$, THESE EQUATIONS ARE

(1') $x_A + x_B + z_2 = 12$

(2') $y_A + y_B = 4 + q_2 = 4 + \frac{1}{2} z_2$.

PARETO EFFICIENCY REQUIRES $MRS_A = MRS_B = f_1'(z_1) = f_2'(z_2)$.

SINCE $f_1'(z_1) = f_2'(z_2) = \frac{1}{2}$, THIS YIELDS

$y_A = \frac{1}{2} x_A$ AND $y_B = \frac{1}{2} x_B$.

THEREFORE (1') AND (2') BECOME

(1'') $x_A + x_B = 12 - z_2$

(2'') $\frac{1}{2}(x_A + x_B) = 4 + \frac{1}{2} z_2$, i.e., $x_A + x_B = 8 + z_2$.

$\therefore 12 - z_2 = 8 + z_2$; i.e., $2z_2 = 4$; $z_2 = 2$, $q_2 = 1$

$\therefore z_1 = 4, q_1 = 4$; $z_2 = 2, q_2 = 1$; $x_A + x_B = 10$, $y_A + y_B = 5$

AND $y_A = \frac{1}{2} x_A$ AND $y_B = \frac{1}{2} x_B$.

(b) Applying the First Welfare Theorem, we know from (a) that $z_1 = 4$, $z_2 = 2$, $y_A = \frac{1}{2}x_A$, and $y_B = \frac{1}{2}x_B$, so that

$$MRS_i = \frac{1}{2} \quad (i = A, B) \quad \text{and} \quad f_j'(z_j) = \frac{1}{2} \quad (j = 1, 2).$$

The market prices must satisfy $p_x = \frac{1}{2} p_y$ in order to elicit these individual decisions. Let's say $p_x = 1$ and $p_y = 2$.

We also have $q_1 = 4$ and $q_2 = 1$, so profits are

$$\pi_1 = p_y q_1 - p_x z_1 = (2)(4) - (1)(4) = 8 - 4 = 4$$

$$\pi_2 = p_y q_2 - p_x z_2 = (2)(1) - (1)(2) = 0.$$

Consumers' wealths/incomes are

$$W_A = p_x \mathring{x}_A + 4\theta_A = 8 + 4\theta_A$$

$$W_B = p_x \mathring{x}_B + 4\theta_B = 8 + 4\theta_B.$$

Consumers' budget constraints are

$$p_x x_i + p_y y_i = 8 + 4\theta_i, \quad \text{i.e.,} \quad x_i + 2y_i = 8 + 4\theta_i,$$

and they choose $y_i = \frac{1}{2}x_i$, i.e., $2y_i = x_i$,

so we have

$$x_i = 4 + 2\theta_i, \quad i = A, B$$

$$y_i = 2 + \theta_i, \quad i = A, B.$$

Note that $x_A + x_B = 8 + 2(\theta_A + \theta_B) = 10$ and $y_A + y_B = 5$, as in (a), and $x_A + x_B + z_1 + z_2 = 16$ and $y_A + y_B = q_1 + q_2$.

(3) max $u^A(x, y_A)$ s.t. $x, y_A, y_B \geqq 0$

AND $u^B(x, y_B) \geqq u_B \quad : \lambda$ ← LAGRANGE MULTIPLIERS

$y_A + y_B \leqq \mathring{y} \quad : \sigma$

FOMC (INTERIOR):

$x: \quad u^A_x = -\lambda u^B_x$, i.e., $u^A_x + \lambda u^B_x = 0$

$y_A: \quad u^A_y = \sigma$

$y_B: \quad 0 = -\lambda u^B_y + \sigma$, i.e., $\lambda u^B_y = \sigma$

$$\therefore \frac{1}{\sigma}\left(u^A_x + \lambda u^B_x\right) = 0$$

$$\frac{u^A_x}{\sigma} + \frac{\lambda u^B_x}{\sigma} = 0$$

$$\frac{u^A_x}{u^A_y} + \frac{\lambda u^B_x}{\lambda u^B_y} = 0$$

$$\frac{u^A_x}{u^A_y} + \frac{u^B_x}{u^B_y} = 0$$

$$MRS_A + MRS_B = 0.$$