Chapter Two

MARKET EQUILIBRIUM: A FIRST APPROACH

To be or not to be, that is the question.
—W. Shakespeare, *Hamlet*

1. The Problem

We shall be considering a number of constructions representing economies in which agents take the terms at which they may transact as independently given. This, of course, is a feature of a perfectly competitive economy. A consequence of this is that part of the environment relevant to the decisions of economic agents consists of the prices of various goods that they take as given. Our main concern will be the description of situations in which the desired actions of economic agents are all mutually compatible and can all be carried out simultaneously, and for which we can prove that for the various economies discussed, there exists a set of prices that will cause agents to make mutually compatible decisions.

In carrying out this program we have imposed a number of restrictions on the matters covered and the degree of generality aimed at. Many of these restrictions are removed in the following chapters. Our first omission will be the rigorous discussion of the choices of economic agents; this will be found in Chapters 3 and 4. Our basic, and often implied, hypothesis is that agents have a complete ordering of points in the space of their possible choices and that, of the choices they can actually make in a given market situation, none is taken if there exists one that is preferred. These ideas are sufficiently familiar to justify invoking them informally before establishing them thoroughly. Our second omission, to be put formally presently, is that we shall be exclusively concerned with situations in which, for each set of prices, there is only one "best" choice for each agent. This will allow us to deal with demand and supply functions rather than with "correspondences" (i.e., situations in which the demand for a good, say at a given set of prices, must be represented by a set rather than by a number). The omission is rectified in Chapters 3–5 and further discussed below.

We start our discussion with a consideration of a very abstract kind of economy and then proceed to analyze a number of more restricted situations.

2. Goods and Prices

A good may be defined by its physical characteristics, its location in space, and the date of its delivery. Goods differing in any of these characteristics will be regarded as different. Services are regarded as goods. We shall suppose here that the number of different goods is finite.

We shall write p_i as the price of the ith good and we define $\mathbf{p}$ as the n-dimensional *row* vector with components p_i $(i = 1, \ldots, n)$. All prices are expressed in fictional unit of account—say "bancors" —and all prices are viewed from the present. Thus if "i" is a given good to be delivered at some location "A" in "T" periods from now, then p_i is the price that must be paid now for delivery of that good at that place at that time. This supposes the existence of all possible futures markets, a highly unrealistic assumption with which we shall dispense at a later stage.

3. The Excess-Demand Functions

Demand and supply decisions are taken by two kinds of agents: *households* and *firms*. The two are distinguished by the property that firms do, and households do not, take production decisions. This distinction is convenient.

Let x_{hi} be the decision of household h with respect to good i. Then, if $x_{hi} < 0$, we shall say that i is a service supplied by household h; when x_{hi} is non-negative, then i will be a good demanded by h, where this concept includes zero demand. We let $\mathbf{x}_h$ represent the n-dimensional *column* vector with components x_{hi}. Summation over households is indicated by omitting the subscript h. Thus, we have

$$x_i = \sum_h x_{hi} \qquad \mathbf{x} = \sum_h \mathbf{x}_h.$$

We write y_{fi} as the decision of firm f with respect to good i. We regard $y_{fi} < 0$ as denoting that i is an input demanded by f, while y_{fi} non-negative means that i is supplied by f where this concept includes zero supply. Also, $\mathbf{y}_f$ is the n-dimensional column vector with components y_{fi}. Summation over firms is indicated by omitting the subscript f. Thus, we have

$$y_i = \sum_f y_{fi} \qquad \mathbf{y} = \sum_f \mathbf{y}_f.$$

If there are any quantities of goods available in the economy before there is any production or market exchange, then we shall take it that these goods are owned by households. We write $\bar{x}_{hi}$ as the amount of good i owned by household h, and note that for good sense this must be a non-negative quantity. As before, $\bar{\mathbf{x}}_h$ is the n-dimensional column vector with components $\bar{x}_{hi}$ and summation over households is indicated by omitting the subscript h.

Market equilibrium is concerned with the compatibility of the decisions of the different firms and households, and therefore we are interested in the difference between the demand for a good and its total supply. The latter is the sum of the production of the good and the quantities of it available before production. Thus, the total supply of good i is $y_i + \bar{x}_i$. We define the excess demand for good i (written z_i) by

$$z_i = x_i - y_i - \bar{x}_i \qquad i = 1, \ldots, n.$$

We write $\mathbf{z}$ for the n-dimensional column vector with components z_i and refer to it as the *excess-demand vector*. Taking $\bar{\mathbf{x}}$ and $\bar{\mathbf{x}}_h$ as given, we regard $\mathbf{z}$ as a function of $\mathbf{p}$. We shall sometimes refer to $z_i < 0$ as an *excess supply* of good i. We put this formally:

ASSUMPTION 1 (F). To any $\mathbf{p}$ there corresponds a unique number $z_i(\mathbf{p})$ called the *excess-demand function* for i and so a unique vector of excess-demand functions $\mathbf{z}(\mathbf{p})$. We have $z_i(\mathbf{p}) = x_i(\mathbf{p}) - \mathbf{y}_i(\mathbf{p}) - \bar{\mathbf{x}}_i$ and call $x_i(\mathbf{p})$ the *demand function* and $y_i(\mathbf{p})$ the *supply function*.[1]

It is quite important to understand why this assumption is indeed restrictive, and we consider a simple example by way of illustration for which F will not hold. Suppose that, given $\mathbf{p}$, there is a unique household response $x_i(\mathbf{p})$. Suppose further that good i is produced by firm f, which produces no other kind of good, while no other firm produces i. Let the firm choose $\mathbf{y}_f$, among all the choices of $\mathbf{y}_f$ open to it, so as to maximize $\mathbf{p}\mathbf{y}_f$, its profits. Assume that $\mathbf{p}$ is such that this maximization is possible and is achieved for $\mathbf{p}\mathbf{y}_f = 0$,

[1] Of course $\mathbf{y}(\mathbf{p})$ is a vector and contains negative components so that this use of the notion "supply function" is not that of the textbook.

but that the firm produces under constant returns to scale. Then evidently for $k > 0$, $k\mathbf{y}_f$ will also maximize profits and so $\mathbf{p}$ does not determine the total supply of good i uniquely and F does not hold. Other examples are possible, none of which relies on unrealistic postulates. It is clear, therefore, that we shall have to regard F as an assumption that, at some stage, we must do without.

There is one other important point that requires emphasis. The number $z_i(\mathbf{p})$ will later be derived from a proper theory of the actions of economic agents, households, and firms. It tells us what the excess demand for i will be if all attempted to carry out their preferred actions at $\mathbf{p}$. The excess-demand function is thus an *ex ante* concept; it is hypothetical in the sense that the actual purchases and sales may differ from those that the theory of the decisions of agents tells us would be the purchases and sales regarded as proper by the agents at $\mathbf{p}$.[1] Indeed, at $z_i(\mathbf{p})$ positive, for instance, it clearly would not be possible for all the agents to complete the transactions with respect to i that they regard as desirable at $\mathbf{p}$.

4. The Main Assumptions

In this section we introduce the main assumptions to be used in this chapter. Many of these will be deduced as propositions from more basic postulates later in this book (Sections 3.4 and 4.5).

The first assumption asserts that the actions of agents depend on the rates at which goods exchange one against another and not at all on the rate at which goods exchange against the (fictional) unit of account, in this case, bancors. This assumption should not be misunderstood. If one of the goods acts as a medium of exchange, for instance, then it too will have a price in terms of unit of account, and it is not asserted that the rate at which goods exchange against this particular good, the medium of exchange, is of no consequence to the decisions of economic agents. We write this assumption formally:

ASSUMPTION 2 (H). $\mathbf{z}(\mathbf{p}) = \mathbf{z}(k\mathbf{p})$ for all $\mathbf{p} > \mathbf{0}$ and $k > 0$; the excess-demand functions are *homogeneous of degree zero in* $\mathbf{p}$.

A consequence of H is that we may fix the level of $\mathbf{p}$ arbitrarily without restricting our analysis in any way. For our purposes, this

[1] See Sections 13.6 and 14.4 for detailed discussions.

is most conveniently done by considering only those prices that belong to the n-dimensional simplex S_n, which is defined by

$$S_n = \left\{\mathbf{p} \,\middle|\, \sum_i^n p_i = 1, \quad \mathbf{p} > \mathbf{0}\right\}.$$

It may be objected that this procedure is rather drastic because it precludes from consideration situations in which all prices are zero and also situations in which some price is negative.

First we note that if we wished to examine an economy from which the "economic problem" of scarcity is absent, we could do so by supposing everyone to own some quantities of a good, the price of which we set equal to unity so that all the other goods have a zero price. This we can do while restricting all $\mathbf{p}$ to be in S_n. The proper representation of such an economy could not be achieved by setting $\mathbf{p} = \mathbf{0}$.

There is also a technical reason for excluding $\mathbf{p} = \mathbf{0}$ from consideration. Suppose H holds for $k \geq 0$ and not just for $k > 0$. Now consider two price vectors, $\mathbf{p}$ and $\mathbf{p}'$, with $\mathbf{z}(\mathbf{p}) \neq \mathbf{z}(\mathbf{p}')$. Then by H, $\mathbf{z}(k\mathbf{p}) = \mathbf{z}(\mathbf{p})$ and $\mathbf{z}(k\mathbf{p}') = \mathbf{z}(\mathbf{p}')$ for $k > 0$. Evidently, if we allow k to approach zero, the two vectors of the excess-demand functions must approach different limits, from which we conclude that $\mathbf{z}(\mathbf{p})$ is not continuous at $\mathbf{p} = \mathbf{0}$. But we certainly would find it very inconvenient to have to do without the continuity of the excess-demand function at any point of the price domain we consider, and in this instance nothing of economic interest would be gained.

The reason for excluding negative prices is less cogent and also less necessary for the subsequent analysis, though we shall maintain it for simplicity. If a good has a negative price, then the individual selling that good has to give up units of some other good or goods with positive price as well. If he also has the option of disposing of the good without giving up any other good with a positive price, it is reasonable to suppose that he will prefer this option. Thus the exclusion of negative prices from consideration is justified if we suppose that there always exists the option of *free disposal*, for then no one would be willing to transact at a negative price and so a negative price could not arise. In this chapter and elsewhere in the book, free disposal is postulated.

The second assumption derives from the fact that, stealing apart, no agent can plan a greater expenditure on goods and services than the receipts he plans to obtain from the sale of goods and services.

By our definition of goods (Section 2.2), this "accounting restraint" includes borrowing and lending. The individual borrows by selling goods now for future delivery and he lends by buying goods now for future delivery.

The difference between the total value of all purchases planned by households and firms and the total value of sales planned by them is evidently $\mathbf{pz}$. We shall now give reasons for taking this number to be non-positive. Suppose that all firms aim to maximize their profits and that they all have the choice of not engaging in any productive activity. Clearly, since $\mathbf{py}$ is the profits of all firms taken together, we may take it that $\mathbf{py}$ is always non-negative. Next suppose that every household h receives a given fraction, $d_h \geq 0$, of the total profits of firms and that

$$\sum_h d_h = 1.$$

Then h may choose any $\mathbf{x}_h$ that satisfies

$$\mathbf{px}_h - \mathbf{p\bar{x}}_h - d_h\mathbf{py} \leq 0. \tag{1}$$

If we can suppose further that a household always prefers $\mathbf{x}_h$ to $\mathbf{x}'_h$ if $\mathbf{x}_h > \mathbf{x}'_h$ and that it will never choose an action if a preferred one is available, then the reader can verify that $\mathbf{x}_h$ will always be such as to make the expression in (1) equal to zero. Summing (1) over h then gives $\mathbf{pz} = 0$.

These are the underlying rationalizations of the assumption that we now put formally.

ASSUMPTION 3 (W). For all $\mathbf{p} \in S_n$, $\mathbf{pz(p)} = 0$ (Walras' law). In what follows we shall need another, rather technical assumption.

ASSUMPTION 4 (C). The vector excess-demand function, $\mathbf{z(p)}$, is continuous over its domain of definition, S_n.

Assumption C implies that $\mathbf{z(p)}$ is bounded everywhere, since S_n is a compact set, which here means that it is closed and bounded. In particular, this means that the demand for a free good is bounded; every individual becomes satiated with respect to any particular good. Unfortunately this assumption comes close to being inconsistent with the reasoning underlying W, which requires that at any point the household is unsatiated with respect to at least one good. A weaker continuity assumption that permits unlimited demand for free goods will be introduced in Section 2.8.

It may help the reader to have an example of the violation of C. Suppose again that good i is produced only by firm f, which produces no other goods. Given the prices of all goods other than i, assume the average cost curve of f to be U shaped.[1] Let p_i^* be the lowest price at which the firm can cover average costs. Then by the usual assumptions, the firm's output will be zero for all $p_i < p_i^*$, while we stipulate that there is a positive output at p_i^*. It can be left to the reader to verify that C will be violated. Since this example is not fanciful, we must conclude that C indeed may be a serious restriction on our analysis. Although this assumption can be somewhat relaxed after F is abandoned, we cannot do without something very close to it in many of the results to be given both in this chapter and in this book. However, we hope to draw some conclusions for the working of an economic system in certain of the cases in which C is violated.

5. Equilibrium

Economic agents may be taken to reach their decisions in the light of what they want and what they can get. If tastes and technology are given, and if the goods owned by individuals and households are also given, then the variables influencing their decisions are the prices prevailing in the various markets. If at some set of admissible prices (i.e., for us some $\mathbf{p}$ in S_n) all these decisions can be carried out simultaneously, then we may say that these decisions are compatible and that the prices are equilibrium prices. There are really two sets of ideas involved in this notion of equilibrium. On the one hand, in such a situation every agent can achieve what he wishes to achieve. On the other hand, if tastes, technology, and the ownership of goods remain given, there will be no mechanism to bring about a change in $\mathbf{p}$. Under the conditions postulated, it is argued that a change in prices is a signal, the consequence of incompatibility in the decisions of agents. This is a familiar notion, the "law of demand and supply," which is discussed more extensively in Chapters 11, 12, and 13.

[1] Since we have taken $\mathbf{p} \in S_n$, the curve must be thought of as constructed as follows: Take any $\mathbf{p} \in S_n$, and suppose that at this $\mathbf{p}$ there is at least one point on i's average cost curve equal to p_i. Now, say, raise p_i to p_i' and multiply all other prices by $k < 1$ so that the new $\mathbf{p}' \in S_n$. Again find a point on the average cost curve at $\mathbf{p}'$ that is equal to p_i'. Proceed in this way to trace the whole curve.

Before we formalize this idea we must take note of a special point. The decision to supply a good in a perfectly competitive economy is not a decision to supply so-and-so much to such-and-such agents, but simply to exchange so-and-so much of the good for other goods. If the price ruling for a good is zero and agents plan to supply some of it, then by the assumption of free disposal (see discussion of H in Section 2.4), we may simply say that agents decide to dispose of a certain amount of the good. Clearly this decision can be carried out by our assumption, whatever the demand of other agents for that good may be. If this demand were greater than the amount offered, however, then the decisions of the demanding agents could not be carried to fruition. From this we conclude that while we would never be willing to regard a situation with positive excess demand in some market as an equilibrium, an excess supply in a market where the price is zero is quite consistent with our notion of an equilibrium. All this seems agreeable to common sense and it remains to put it more formally.

DEFINITION 1 (E). $\mathbf{p}^*$ in S_n is called an *equilibrium* if $\mathbf{z}(\mathbf{p}^*) \leq \mathbf{0}$, where $\mathbf{z}(\mathbf{p})$ is derived from the "preferred" actions of agents.

That this formal definition indeed corresponds to our discussion of the equilibrium concept can be seen with the aid of the following theorem.

THEOREM 1. If W and $\mathbf{z}(\mathbf{p}^*) \leq \mathbf{0}$, then $z_i(\mathbf{p}^*) < 0$ implies that $p_i^* = 0$.

Proof. Since $\mathbf{p}^*$ is in S_n, it follows from the assumptions of T.2.1 that every element in the sum $\mathbf{p}^*z(\mathbf{p}^*)$ is non-positive. Then, if, contrary to what is asserted, $p_i^* > 0$, it must be that $\mathbf{p}^*z(\mathbf{p}^*) < 0$, which contradicts W. Since no price can be negative, this completes the proof.

In several respects D.2.1 is incomplete because it does not specify the conditions that must hold for each agent if his decision is to be the "best" open to him at $\mathbf{p}^*$. This, however, must be postponed until Chapters 3 and 4. Here we must be satisfied with our rather informal treatment.

It should also be noted that there is no reason to suppose that there is only one equilibrium price vector. The question of the "uniqueness" of an equilibrium will be fully explored in Chapter 9.

Here we simply introduce a piece of notation: We write E for the set of equilibrium price vectors,

$$E = \{\mathbf{p} \mid \mathbf{z}(\mathbf{p}) \leq \mathbf{0}; \quad \mathbf{p} \in S_n\}.$$

6. The Existence of Equilibrium—the Case of Two Goods

This section is to serve as an introduction to the proof that in general, given our assumptions, the set E is not empty. It is hoped that it will facilitate a proper appreciation of the roles of the various assumptions in the proof. In what follows we take all $\mathbf{p}$ to be in S_n.

Consider, in a two-good economy, the two price vectors

$$\mathbf{p}' = (0,1) \quad \text{and} \quad \mathbf{p}'' = (1,0).$$

We suppose that neither $\mathbf{p}'$ nor $\mathbf{p}''$ is in E, else there is nothing to prove. But then it must be that $z_1(\mathbf{p}') > 0$ and $z_2(\mathbf{p}'') > 0$. Consider the first of these. By W we have $0z_1(\mathbf{p}') + 1z_2(\mathbf{p}') = 0$. If, contrary to our assertion, we had $z_1(\mathbf{p}') \leq 0$, then the first term would certainly be zero and so also $z_2(\mathbf{p}') = 0$, which contradicts the supposition that $\mathbf{p}'$ is not in E. The same argument establishes the inequality for $z_2(\mathbf{p}'')$.

Now let

$$\mathbf{p}(m) = m\mathbf{p}' + (1 - m)\mathbf{p}'' \qquad \text{with } 1 \geq m \geq 0.$$

By W, one of the numbers $z_1(\mathbf{p}(m))$ and $z_2(\mathbf{p}(m))$ is positive and the other negative when $m \neq 0,1$, unless $\mathbf{p}(m) \in E$. Suppose then, without loss of generality, that $z_1(\mathbf{p}(m^0)) < 0$ for some $m^0 \neq 0,1$. Now let m increase from m^0 to 1. We already know that $z_1(\mathbf{p}(0))$ is positive, and therefore, as m approaches zero, somewhere $z_1(\mathbf{p}(m))$ must change sign. But, by C, it cannot change sign without becoming zero. Suppose this happens at m^*. Then $\mathbf{p}(m^*)$ is in E, for, by W, it must be that $z_2(\mathbf{p}(m^*)) = 0$.

We note the important role of C in this demonstration. Without it we could not exclude the possibility of a change in the sign of z_1, as m approaches zero, without its ever becoming equal to zero. Of course we also relied heavily on W, but this assumption does not appear to be very restrictive. As was indicated in the discussion of C, however, the latter will certainly exclude a number of perfectly possible situations from consideration. Later in the book, an investigation of some of these possibilities should help us to form some judgment as to the likelihood that the coherence of decisions implied by equilibrium is attainable by actual economies. In this, however, due care will have to be taken not to confuse the statement "an equilibrium cannot be shown to exist" with the statement "no equilibrium is possible."

In Figure 2-1 we illustrate the proposition just established for a two-good economy. In the diagram the horizontal axis is of unit length.

7. The Existence of an Equilibrium: Many Goods

When we turn to the economy with many goods, it is clear that the simple procedure of Section 2.6 will not serve, although the lessons we have learned will continue to be of interest, as we shall see. Indeed, the best introduction to the general case is probably achieved by staying with the two-good case a little longer.

Take any arbitrary point **p** on the horizontal axis of Figure 2-1 so that **p** is in S_n. At the point chosen in the figure, $z_1(\mathbf{p})$ is positive and $z_2(\mathbf{p})$ is negative. Let us adopt the following rules:

(1) Raise the price of the good in positive excess demand.

(2) Lower or at least do not raise the price of the good in excess supply, but never lower the price below zero.

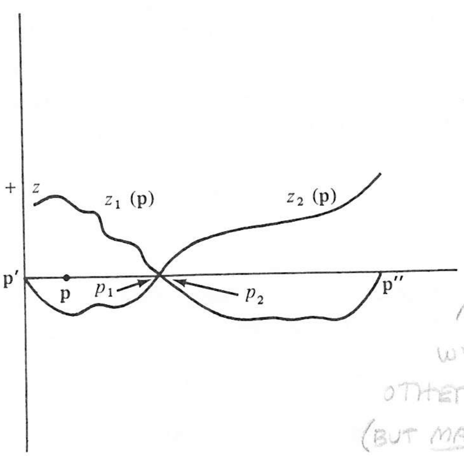
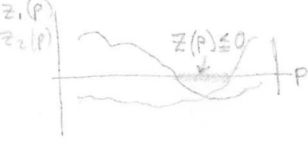


Figure 2-1

IF WE RELAX W TO $pz(p) \leq 0$, THEN WE ONLY REQUIRE THAT BOTH LINES NEVER BE ABOVE AXIS AT SAME TIME; THEN, WHEN ONE CROSSES AXIS, OTHER MUST NOT BE ABOVE IT (BUT MAY BE BELOW, AUTOMATICALLY GIVING — BY CONTINUITY — AN INTERVAL OF EQUIL'M p's.)



(3) Do not change the price of a good in zero excess demand.
(4) Multiply the resulting price vector by a scalar, leaving relative prices unchanged, so that the new price vector you obtain is in S_n.

If we are successful in carrying out these rules we can say: Given any **p** in S_n, we have a routine for finding another point in S_n. Another way of putting this is to say that our procedure gives us a *mapping* of S_n into itself. We note that if **p** is an equilibrium, then the mapping will give us **p** again. The converse will also be true: If the mapping takes us from **p** back to **p**, then equilibrium exists. The question then is: Does at least one such point exist? In our two-good example the answer is clearly "yes." Suppose that neither $\mathbf{p}'$ nor $\mathbf{p}''$ is the point we seek. We know that at $\mathbf{p}'$ the rules tell us to raise the price of good 1 and at $\mathbf{p}''$ they tell us to raise the price of good 2. By W, though, we can never be asked to raise the prices of both goods at any **p**. Hence, at $\mathbf{p}(m)$ with $1 \neq m \neq 0$, the rules instruct us to lower the price of some good, say the first. But then by C at some $\mathbf{p}(m^*)$, the rules tell us not to change the price of the first good (because $z_1(\mathbf{p}(m^*)) = 0$), and then by W it follows also that we must not change the price of the second good. Hence, $\mathbf{p}(m^*)$ is a point of the kind we seek.

All this is really a repetition of the argument of the previous section in slightly different terms. When we come to the case of many goods, our method will have to be somewhat different. As much as possible, we shall use the economics of our problem to construct a procedure that satisfies the rules we have given. We can use C to establish that the rules give a continuous mapping. Then we will appeal to a mathematical theorem that assures us that there will be at least one point in S_n that the mapping returns to itself. We then will appeal again to our economics to show that this point is the equilibrium we seek.

Step 1: Construction of mapping. We first seek a continuous function with the following three properties:

$$M_i(\mathbf{p}) > 0 \quad \text{if and only if } z_i(\mathbf{p}) > 0, \tag{2a}$$

$$M_i(\mathbf{p}) = 0 \quad \text{if } z_i(\mathbf{p}) = 0, \tag{2b}$$

$$p_i + M_i(\mathbf{p}) \geq 0. \tag{2c}$$

It is intended that $M_i(\mathbf{p})$ represent an adjustment to an existing price so that a price vector $\mathbf{p}$ is transformed into a new price vector with components $p_i + M_i(\mathbf{p})$.

Functions satisfying (2) exist; for one example, let $M_i(\mathbf{p}) = \max(-p_i, k_i z_i(\mathbf{p}))$, where $k_i > 0$. Since $M_i(\mathbf{p})$ is a continuous transformation of p_i and z_i, it is certainly continuous by C.

To verify (2a), first suppose $z_i(\mathbf{p}) > 0$. Since $p_i \geq 0$, $k_i z_i(\mathbf{p}) > -p_i$, so that $M_i(\mathbf{p}) = k_i z_i(\mathbf{p}) > 0$. Conversely, if $z_i(\mathbf{p}) \leq 0$, then either $M_i(\mathbf{p}) = k_i z_i(\mathbf{p}) \leq 0$ or $M_i(\mathbf{p}) = -p_i \leq 0$. It is easy to verify that (2b) and (2c) hold.

An even simpler, though less intuitive, example of a function satisfying (2) is $M_i(\mathbf{p}) = \max(0, k_i z_i(\mathbf{p}))$, $k_i > 0$.

For functions satisfying (2), we easily deduce

$$M_i(\mathbf{p}) z_i(\mathbf{p}) \geq 0 \qquad \text{all } i. \tag{3}$$

It will be seen that if we interpret $p_i + M_i(\mathbf{p})$ as the ith component of the new price vector that the mapping produces, given $\mathbf{p}$, the procedure for finding these new prices satisfies rules discussed earlier. However, while all $p_i + M_i(\mathbf{p})$ are certainly non-negative, there is nothing to ensure that they will add up to one. In other words, if we write $\mathbf{p} + \mathbf{M}(\mathbf{p})$ as the row vector of the new prices (components $p_i + M_i(\mathbf{p})$), then there is no reason to suppose that $\mathbf{p} + M(\mathbf{p})$ is in S_n when $\mathbf{p}$ is in S_n. Since we seek a mapping of S_n into itself, we must modify the mapping.

An obvious way of doing this is as follows: Let $\mathbf{e}$ be the n-dimensional column vector with all components unity. Then $[\mathbf{p} + M(\mathbf{p})]\mathbf{e}$ is certainly non-negative. If we are certain that this number is strictly positive, then we may take the mapping given by

$$\mathbf{T}(\mathbf{p}) = \frac{\mathbf{p} + \mathbf{M}(\mathbf{p})}{[\mathbf{p} + \mathbf{M}(\mathbf{p})]\mathbf{e}}. \tag{4}$$

The reader can now verify that (4) obeys all of the four rules we have laid down, in particular, that the vector $\mathbf{T}(\mathbf{p})$ is in S_n.

We now show that (4) is indeed a possible mapping by proving that for all $\mathbf{p} \in S_n$, $[\mathbf{p} + \mathbf{M}(\mathbf{p})]\mathbf{e} > 0$. If not, then for some $\mathbf{p} \in S_n$, $\mathbf{p} + \mathbf{M}(\mathbf{p}) = \mathbf{0}$ by 2(c). But then

$$0 = [\mathbf{p} + \mathbf{M}(\mathbf{p})]\mathbf{z}(\mathbf{p}) = \mathbf{p}\mathbf{z}(\mathbf{p}) + \mathbf{M}(\mathbf{p})\mathbf{z}(\mathbf{p}) = \mathbf{M}(\mathbf{p})\mathbf{z}(\mathbf{p})$$

by W. But then by (3), $M_i(\mathbf{p}) z_i(\mathbf{p}) = 0$, all i. Since $\mathbf{p} \in S_n$, it must be that $p_i > 0$, some i, so for that i, $M_i(\mathbf{p}) = -p_i < 0$. This, however, must mean $z_i(\mathbf{p}) = 0$, which in turn by (2b) implies that $M_i(\mathbf{p}) = 0$, a contradiction. Hence, $[\mathbf{p} + \mathbf{M}(\mathbf{p})]\mathbf{e} > 0$ for all $\mathbf{p} \in S_n$.

Step 2: The mathematical result. We first define some of the notions of our introductory remarks more formally.

DEFINITION 2. (a) If $\mathbf{T}(\mathbf{p})$ is a mapping that takes points in S_n into points in S_n, then the mapping is said to map S_n into itself.

(b) If for some $\mathbf{p}^*$ we have $\mathbf{p}^* = \mathbf{T}(\mathbf{p}^*)$, then $\mathbf{p}^*$ is called a *fixed point* of $\mathbf{T}(\mathbf{p})$.

The theorem we use in this chapter is called Brouwer's fixed-point theorem, and it is stated as follows: Every continuous mapping of a compact convex set into itself has a fixed point. The proof of this result will be found in T.C.1; convexity is defined in D.B.7. Here we need confirm only that the set S_n that we are interested in satisfies the requirements of the theorem.

Certainly if $\mathbf{p}$ and $\mathbf{p}'$ are in S_n, then for any m with $1 \geq m \geq 0$, the vector $\mathbf{p}(m) = m\mathbf{p} + (1 - m)\mathbf{p}'$ is in S_n, since $\mathbf{p}(m)$ is non-negative and $\mathbf{p}(m)\mathbf{e} = 1$. Hence, S_n is convex. Since S_n is clearly bounded and since the limit point of any sequence of price vectors in S_n is itself in S_n, S_n is compact.

Step 3: The fixed point of $\mathbf{T}(\mathbf{p})$ is an equilibrium. At the fixed point, we have $\mathbf{p}^* = \mathbf{T}(\mathbf{p}^*)$, that is,

$$\{[\mathbf{p}^* + \mathbf{M}(\mathbf{p}^*)]\mathbf{e}\}\mathbf{p}^* = \mathbf{p}^* + \mathbf{M}(\mathbf{p}^*)$$

or

$$\mathbf{M}(\mathbf{p}^*) = \lambda\mathbf{p}^*, \tag{5}$$

where $\lambda = [\mathbf{p}^* + \mathbf{M}(\mathbf{p}^*)]\mathbf{e} - 1$. Take the inner product of (5) on both sides with $\mathbf{z}(\mathbf{p}^*)$, and use W to find

$$\mathbf{M}(\mathbf{p}^*)\mathbf{z}(\mathbf{p}^*) = \lambda\mathbf{p}^*\mathbf{z}(\mathbf{p}^*) = 0.$$

As before, $M_i(\mathbf{p}^*)z_i(\mathbf{p}^*) = 0$, all i, by (3). Hence, $z_i(\mathbf{p}^*) > 0$ would imply $M_i(\mathbf{p}^*) = 0$, which contradicts (2a). Hence, $z_i(\mathbf{p}^*) \leq 0$, all i. By D.2.1, $\mathbf{p}^*$ is an equilibrium.

We summarize the result of this section formally:

THEOREM 2. If F, H, W, and C, then an equilibrium for a competitive economy with a finite number of goods exists.