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Arrow-Debreu Equilibrium

The Arrow-Debreu model incorporates time and uncertainty in general equilibrium theory in a way that preserves the theory's main conclusions but at the cost of making the unrealistic assumption that all trading takes place at one initial moment, before anything other than trade occurs. Time and uncertainty are included by collapsing the entire future into a fictitious present. Despite this drawback, the model enriches the theory and its interpretation, for the extreme restriction on trading may be replaced by more plausible assumptions.

7.1 The Arrow-Debreu Model

The Arrow-Debreu approach requires a model of uncertainty and the passage of time. Uncertainty is represented by using a set of states, S, called the *states of the world.* I usually assume that S is a finite set, though a valid theory exists with an infinite set S. Each state s in S is a complete description of everything that is relevant to the situation studied. It should be imagined that only one state, s, actually occurs. What people observe are *events,* and these are subsets of S. Not all subsets of S are included as events, but only those that are considered to be observable. It is assumed that if A and B are events, then $A \cup B$, $A \cap B$, and $S \backslash A = \{s \in S \mid s \notin A\}$ are events as well. The set of events may or may not include subsets consisting of a single state. It is assumed that the whole set, S, and the empty set, $\emptyset$, are events. Probabilities are assigned to events by means of a function p, from the set of events to the unit interval, $[0, 1]$. If A is an event, $p(A)$ is, roughly speaking, the proportion of times that A would occur if the circumstances generating the observation were repeated a great many times. If $p(A) = 1$, then A

would always occur, and if $p(A) = 0$, A would never occur. It is assumed that $p(S) = 1$ and that $p(A \cup B) = p(A) + p(B)$, if A and B are disjoint events. These two assumptions imply that $p(\emptyset) = 0$. If S is a finite set, then the probability of an event is the sum of the probabilities of the individual states in the event, provided these states are themselves events. If there are infinitely many states, it may not be possible to build up the probabilities of all events from those of individual states. The following examples illustrate these notions.

EXAMPLE 7.1 You toss a fair coin twice. The set of states is

$$S = \{(H, H), (H, T), (T, H), (T, T)\},$$

where H represents heads and T represents tails. The state (H, T) represents heads on the first toss and tails on the second. Each state, (x, y), may be thought of as an event $\{(x, y)\}$, where x and y can be either H or T. The probability of each state is 1/4, so that $p((H, H)) = 1/4$ and so on. The set of events is the set of all possible subsets of S. The event of heads on the second toss is $\{(H, H), (T, H)\}$, and the probability of this event is $p((H, H)) + p((T, H)) = 1/4 + 1/4 = 1/2$.

EXAMPLE 7.2 A coin is tossed twice by an unknown mechanism, and you are told the number of times heads comes up. The set of states is as in the previous example, but the observable events are only the empty set, $\emptyset$, the whole set, S, and the sets $\{(T, T)\}$, $\{(H, T), (T, H)\}$, and $\{(H, H)\}$. It makes sense to assign probabilities only to these observable sets, since the mechanism generating the tosses is unknown.

EXAMPLE 7.3 A number is chosen from the interval $[0, 1]$ with uniform probability. The set of states is $[0, 1]$. Events are what is known as Lebesgue measurable subsets of $[0, 1]$. It includes all intervals, such as $[a, b]$, as well as many other sets, but does not include all subsets of $[0, 1]$. The probability of any single state, $\{s\}$ is zero, whereas the probability of the interval $[a, b]$ is $b - a$, if $b \geq a$. If $b > a$, the probability of $[a, b]$ cannot, therefore, be the sum of the probabilities of the individual states in $[a, b]$.

In example 7.1, events occur in a temporal sequence; there is a first toss, and there is a second toss. The set of states is the set of histories of what occurs. These histories may be described by a tree diagram, as in figure

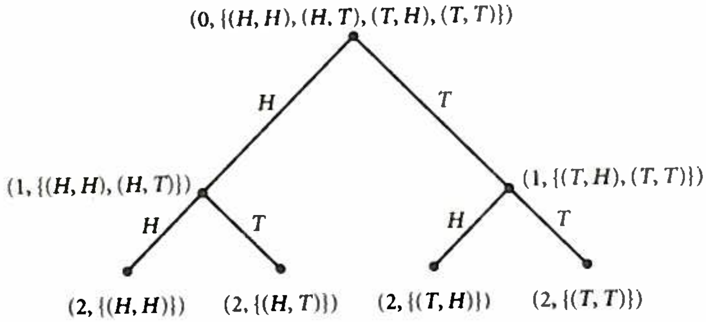

Figure 7.1 A dated event tree

7.1. The nodes in the figure correspond to what are called *dated events.* The dated event $(0, \{(H, H), (H, T), (T, H), (T, T)\})$ is the situation at time 0 before anything has happened. The dated event $(1, \{(T, H), (T, T)\})$ is the situation at time 1 occurring after tails occurred at time 0. The dated event $(2, \{(H, T)\})$ is the situation at time 2 after heads occurred at time 0 and tails at time 1.

I now describe terminology and notation that permits discussion of general dated events.

DEFINITION 7.4 A *partition* of a set S is a set, $\mathcal{S}$, of nonempty subsets of S that are mutually disjoint and whose union is S. That is, $A \cap B = \emptyset$, for any distinct members, A and B, of $\mathcal{S}$, and $\cup_{A \in \mathcal{S}} A = S$.

DEFINITION 7.5 If $\mathcal{F}$ and $\mathcal{S}$ are partitions of S, $\mathcal{F}$ *refines* $\mathcal{S}$ if every A in $\mathcal{S}$ is a union ofsets in $\mathcal{F}$. That is, for every A in $\mathcal{S}$, the sets in $\mathcal{F}$ that are subsets of A form a partition of A.

Partitions can be used to express the revelation of information over time. Suppose that the information is revealed over periods $t = 0, 1, \ldots, T$ and that S is the set of possible states of the world. The amount of information available at time t is represented by a partition, $\mathcal{S}_1$, of S. If nothing is forgotten, so that information increases over time, then $\mathcal{S}_{t+1}$ refines $\mathcal{S}_t$, for all t. The partition $\mathcal{S}_t$ is the set of events that occur up to time t. Suppose that such a sequence of partitions, $\mathcal{S}_t$, is given, for $t = 0, 1, \ldots, T$.

DEFINITION 7.6 The *set of dated events*, Γ, equals $\{(t, A) \mid 0 \le t \le T, A \in \mathcal{S}$, for all $t\}$, where in a pair (t, A) the letter t is the date of occurrence of the event A.

There are as many dated events for period t as there are events in $\mathcal{S}_t$. Since $\mathcal{S}_0$ may contain many events, there may be many dated events for period 0.

Imagine that all the agents in an economy observe the same events, and let the dated event set, Γ, be as above. Finally, imagine that the same N commodities are available in each dated event.

DEFINITION 7.7 A *contingent claim* is an agreement to deliver or receive an amount of a specified commodity in a specified dated event.

The set of all vectors of quantities of commodities in dated events is

$$\mathbb{R}^{\Gamma \times N} = \{\mathbf{x}: \Gamma \times \{1, \ldots, N\} \to \mathbb{R}\} = \{\mathbf{x}: \Gamma \to \mathbb{R}^N\},$$

where $\Gamma \times N$ denotes the Cartesian product of the sets Γ and $\{1, 2, \ldots, N\}$. That is, $\Gamma \times N = \{((t, A), n) \mid (t, A) \in \Gamma$ and $n = 1, 2, \ldots, N\}$. (The *Cartesian product* of sets A and B is $A \times B = \{(a, b) \mid a \in A$ and $b \in B\}$.) A typical component of a vector $\mathbf{x}$ in $\mathbb{R}^{\Gamma \times N}$ is $x_{(t, A), n}$, where (t, A) is a dated event in Γ and n is one of the N commodities. If S is finite, the sets Γ and $\Gamma \times N$ are finite as well.

Imagine that trade in all the contingent claims occurs at a moment, time -1, just preceding time 0, that trade is made against a single unit of account, and that no trade occurs after the initial moment when trade in contingent claims occurs. In periods $0, 1, \ldots, T$, deliveries are made and taken according to the contingent contracts purchased and sold at time -1. Trade occurs at time -1 for commodities in all dated events of any given period t, even though only one of those events is actually realized.

Imagine an economy where all trading is in contingent claims. In such an economy, if you wanted to buy 5 pounds of wild bird food on a certain winter day (but only if there had been snow on the ground for at least a week), the purchase would be arranged beforehand on a market for bird food, at that date and when there had been snow on the ground for at least a week. An economy where all trades are arranged through forward contingent trades is said to be *Arrow-Debreu* or to have *complete markets*. To model such an economy, let the input-output possibility set, Y_j of the

jth firm be a subset of $\mathbb{R}^{\Gamma \times N}$, for each j, and for each consumer i, let the utility function be $u_i: \mathbb{R}_+^{\Gamma \times N} \to \mathbb{R}$ and the endowment be $\mathbf{e}_i \in \mathbb{R}_+^{\Gamma \times N}$, where $\mathbb{R}_+^{\Gamma \times N}$ is the set of all vectors in $\mathbb{R}^{\Gamma \times N}$ with nonnegative components. An equilibrium in such a model is termed an *Arrow-Debreu equilibrium.* It is denoted by $(\mathbf{x}, \mathbf{y}, \mathbf{p})$, where $(\mathbf{x}, \mathbf{y})$ is a feasible allocation and $\mathbf{p} \in \mathbb{R}_+^{\Gamma \times N}$ is a price vector. We know from the first welfare theorem 5.2 (in section 5.1) that if the utility functions are locally nonsatiated, then any equilibrium allocation is Pareto optimal. The Pareto optimality of Arrow-Debreu equilibria explains economists' interest in them.

The following example should clarify the meaning of the concept.

EXAMPLE 7.8 There are two periods, 0 and 1, two states in the second period, a and b, and the events are $\{a\}$ and $\{b\}$. The dated events are $(0, \{a, b\})$, $(1, \{a\})$, and $(1, \{b\})$. Let $\{a, b\} = S$. There is one firm, one consumer, and one commodity in each dated event, so that this is a Robinson Crusoe economy. The consumer is endowed with one unit of the good in period 0 and none in period 1 in either state. That is, the consumer's initial endowment is

$$\mathbf{e} = (e_{(0,S)}, e_{(1,a)}, e_{(1,b)}) = (1, 0, 0),$$

where I use $(1, a)$ and $(1, b)$ to stand for $(1, \{a\})$ and $(1, \{b\})$, respectively. Each of the two states a and b has probability one-half, the consumer has utility $\ln(x)$ for consumption of x units of the good in any state, and the consumer's utility is the sum of the utility from consumption in period 0 and the expected utility from consumption in period 1. That is, the consumer's utility function is

$$u(x_{(0,S)}, x_{(1,a)}, x_{(1,b)}) = \ln(x_{(0,S)}) + \frac{1}{2}\ln(x_{(1,a)}) + \frac{1}{2}\ln(x_{(1,b)}).$$

The firm's output is $\sqrt{-y_{(0,S)}}$ in dated event $(1, a)$, and $-y_{(0,S)}$ in dated event $(1, b)$, where $-y_{(0,S)}$ is the firm's input of the good in period 0.

Since the Arrow-Debreu equilibrium allocation in this example is Pareto optimal, it is optimal, so that the equilibrium allocation is the optimal one. That is, in order to calculate the equilibrium allocation, we must solve the problem

$$\max_{x_{(0,S)},x_{(1,a)},x_{(1,b)},y_{(0,S)}} \left[\ln(x_{(0,S)}) + \frac{1}{2}\ln(x_{(1,a)}) + \frac{1}{2}\ln(x_{(1,b)})\right]$$

$$\text{s.t.} \quad y_{(0,S)} \leq o,$$
$$0 \leq x_{(0,S)} \leq 1 + y_{(0,S)},$$
$$0 \leq x_{(1,a)} \leq \sqrt{-y_{(0,S)}}, \text{ and}$$
$$0 \leq x_{(1,b)} \leq -y_{(0,S)}.$$

Solving this problem, we find that $x_{(0,S)} = \frac{4}{7}$, $x_{(1,a)} = \sqrt{\frac{3}{7}}$, $x_{(1,b)} = \frac{3}{7}$, and $y_{(0,S)} = -\frac{3}{7}$. In calculating Arrow-Debreu prices, I normalize them so that $p_{(0,S)} = 1$. A first-order condition for the consumer's utility-maximization problem over a budget constraint is

$$\frac{\partial u(x_{(0,S)}, x_{(1,a)}, x_{(1,b)})}{\partial x_{(0,S)}} = \lambda p_{(0,S)},$$

where λ is the consumer's marginal utility of unit of account. This equation implies that

$$\lambda = \frac{1}{x_{(0,S)}} = \frac{7}{4}.$$

Another first-order condition for the consumer's maximization problem is

$$\frac{\partial u(x_{(0,S)}, x_{(1,a)}, x_{(1,b)})}{\partial x_{(1,a)}} = \lambda p_{(1,a)},$$

which implies that

$$\frac{1}{2}\frac{1}{x_{(1,a)}} = \lambda p_{(1,a)},$$

and hence

$$p_{(1,a)} = \frac{2\sqrt{221}}{21}.$$

Similarly, the first-order condition

$$\frac{\partial u(x_{(0,S)}, x_{(1,a)}, x_{(1,b)})}{\partial x_{(1,b)}} = \lambda p_{(1,b)},$$

implies that

$$\frac{1}{2}\frac{1}{x_{(1,b)}} = \partial p_{(1,b)},$$

and hence that

$$p_{(1,b)} = \frac{2}{3}.$$

There are two important things to notice about this example. First of all, the prices for the good in the dated events $(1, a)$ and $(1, b)$ are not proportional to the events' probabilities, though the consumer maximizes the expected utility of consumption in period 2. The probabilities of the two events are equal, yet $p_{(1,a)} = \frac{2}{\sqrt{21}} < \frac{2}{3} = p_{(1,b)}$. Because more of the good is available in dated event $(1, a)$ than $(1, b)$, the good is cheaper in event $(1, a)$. The second thing to notice is that the firm does not need to know the probabilities of the events in order to maximize its profits. It need only know the Arrow-Debreu prices. The firm's profit-maximization problem is

$$\max_{y_{(0,S)} \geq 0} \left[p_{(0,S)} y_{(0,S)} + p_{(1,a)}\sqrt{-y_{(0,S)}} + p_{(1,b)}(-y_{(0,S)})\right]$$

$$= \max_{y_{(0,S)} \geq 0} \left[y_{(0,S)} + \frac{2}{\sqrt{21}}\sqrt{-y_{(0,S)}} + \frac{2}{3}(-y_{(0,S)})\right].$$

The probabilities appear nowhere in this expression. This feature of the model does not correspond to reality, because executives of actual firms are preoccupied with trying to predict the future. They cannot, however, buy their inputs and sell their outputs on markets for contingent claims. If they could do so, they would, no doubt, care little about the likelihoods of the various future events.

7.2 Arrow Equilibrium

I now present an idea of Kenneth Arrow (1953) that makes Pareto optimality of equilibrium with uncertainty seem somewhat more feasible than it might otherwise appear. The Arrow-Debreu equilibrium strains our credulity, for not only does it require that all trading occur at an initial moment, but people must trade on an enormous number of markets at that time. Suppose that $\mathcal{S}_0 = \{S\}$ and that for every time period t, every member of the set of events, $\mathcal{S}_t$, contains two members of the partition $\mathcal{S}_{t+1}$. Suppose also that there are N goods in each dated event and that $t = 0, 1, \ldots, T$. Then there are N types of contingent claim for period 0, $2N$ for period 1, $4N$ for period 2, and $2^t N$ for period t. Since the number of events in