Economies with Public Goods: An Elementary Geometric Exposition

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Abstract

This paper explains how to represent economies with one private good, one public good, and two agents when the public good is produced from the private good by operating a linear technology, by means of the so-called "Kolm triangle." It also shows the usefulness of this representation in analyzing this class of economies.

1. Introduction

The "Kolm triangle" (Kolm 1970) is a geometric device that provides a representation of two-agent economies with one private good and one public good when the public good is produced from the private good by operating a linear technology. Our purpose is to explain how to construct it and to show that it can be used to illustrate very simply a number of the basic concepts of the theory of public goods, and to obtain important insights into the compatibility of properties of allocation rules in this context.

In the initial sections of this article, Sections 2–6, we show how the points of an equilateral triangle can be put in one-to-one correspondence with the feasible allocations of an economy of the kind in which we are interested. We describe the transformation to which preferences should be subjected for equilateral triangles to be so used. We identify the set of Pareto efficient allocations and derive the marginal conditions for effi-

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Received 23 March 1998; Accepted 20 August 1998.

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Journal of Public Economic Theory, 1 (1), 1999, pp. 139–176.

ciency. We define and illustrate several bounds on welfares that are commonly imposed, and the central notion of a Lindahl equilibrium. We conclude with a list of the mistakes that are made most frequently in the use of the triangle.

In Sections 7–11, which constitute the main justification for having undertaken this exposition, we show how a number of results concerning the existence of efficient allocation rules that satisfy one or the other of the welfare bounds and pass additional tests of good behavior, in particular implementability and monotonicity tests, can be illustrated very simply in the Kolm triangle. When the intuition is well understood in the special case that the triangle represents, the general proof is often not far behind.

Many concepts and results that are easily described or suggested with the help of the Edgeworth box have counterparts for economies with public goods that can be explained equally well by means of the Kolm triangle. However, when a counterpart exists, some minimal adaptation is almost always needed. Errors commited with the Kolm triangle are often the result of overlooking where and how the two models differ. When some important difference exists, we have therefore inserted a comment for the private good case within square brackets.

2. Constructing the Kolm Triangle

A (pure) public good, when produced at a certain level, is consumed at that level by all agents. We consider economies with two goods, one private good and one public good, and two agents, agent 1 and agent 2. Let $N = \{1,2\}$. For each $i \in N$, $x_i \in \mathbb{R}_+$ designates agent i's consumption of the private good. The variable y represents the two agents' common consumption of the public good. Agent $i \in N$ is endowed with an amount ω_{ix} of the private good. The initial level of the public good is 0. Let $\omega_i = (\omega_{ix},0)$ denote agent i's initial bundle. The public good is produced from the private good by operating a linear technology. We choose the units of measurement of the goods so that one unit of the private good, used as input in production, yields one unit of the public good. Then, a (feasible) allocation is a triple $z = (x_1, x_2, y) \in \mathbb{R}^3_+$ such that $x_1 + x_2 + y = \omega_{1x} + \omega_{2x}$. This equation states that the aggregate endowment of the private good, $\omega_{1x} + \omega_{2x}$, is partly consumed as such—accounting for $x_1 + x_2$ units of it—and partly used in the production of the public good—accounting for y units. We refer to the triple $(\omega_{1x},\omega_{2x},0) \in \mathbb{R}^3_+$ as the **initial allocation**. Agent i's consumption bundle at $z = (x_1, x_2, y) \in Z$, his "component of z," is the pair $z_i = (x_i, y)$. Let Z denote the set of allocations:

$$Z = \{(x_1, x_2, y) \in \mathbb{R}^3_+ : x_1 + x_2 + y = \omega_{1x} + \omega_{2x}\}.$$

We will have occasion to refer to excludable public goods. Such a good is consumed by each agent at a level that is *at most equal* to the level at which it is produced.

To complete the description of an economy, we need to specify how agents compare consumption bundles. For each $i \in N$, let R_i denote agent i's preference relation, P_i the strict preference relation associated with R_i, and I_i the corresponding indifference relation. We assume throughout that R_i, in addition to being continuous, is monotone increasing in the sense that for all $z_i, z'_i \in \mathbb{R}^2_+$, if $z'_i > z_i$, then $z'_i P_i z_i$.[1] In most of our figures, preferences are convex (for all $z_i \in \mathbb{R}^2_+$, $\{z'_i \in \mathbb{R}^2_+ : z'_i \; R_i \; z_i\}$ is a convex set). Preferences that are continuous, monotone increasing, and convex are **classical**. Linear, Leontieff, and Cobb–Douglas preferences are all classical. Sometimes we consider strictly monotone preferences (for all z_i, $z'_i \in \mathbb{R}^2_+$, if $z'_i \geq z_i$ then $z'_i \; P_i \; z_i$). We will occasionally abuse language and speak of "an agent preferring some allocation to some other allocation" whenever he prefers his component of the former to his component of the latter.

The Kolm representation is made possible by the fact that in an equilateral triangle the sum of the distances of a point to the three sides is independent of the point.[2] See Figure 1a for instructions on how to read the coordinates of an allocation $z = (x_1, x_2, y)$ in the triangle. The possibility of exclusion is illustrated in Figure 1b. In most cases, a point in the triangle is interpreted as an allocation. On occasions, however, we will want to point to a consumption bundle for a specific individual. We will then use an agent's subscript: in Figure 1b, $\bar{z}_1$ is a bundle for agent 1. On the other hand, numerical superscripts refer to allocations. For instance, z^1 will denote an allocation (whose components are z^1_1 and z^1_2).

To be able to map an economy to a triangle, we have to understand what preferences look like in slanted axes. The passage from rectangular to slanted axes is illustrated in Figure 2. The image in slanted axes of a typical bundle z_i of rectangular coordinates (x_i, y) is denoted by z'_i. It lies at a distance x_i from agent i's slanted axis, and it has the same ordinate as z_i. Its coordinates (x'_i, y') in the original axes are given by $x'_i = 1/\sqrt{3}(2x_i + y)$ and $y' = y$.

The transformation that takes z_i to z'_i is linear. Important consequences of this fact are that the image of a straight line is a straight line, the images of parallel lines are parallel lines, the images of two curves that are asymptotic to each other are two curves that are asymptotic to each other, and the

[1] Vector inequalities: given a, $b \in \mathbb{R}^m$, $a \geqq b$ means that $a_\ell \geq b_\ell$ for all ℓ; $a \geq b$ means that $a \geqq b$ but $a \neq b$; and $a > b$ means that $a_\ell > b_\ell$ for all ℓ.

[2] To prove this result, given a point z in the triangle, simply write that the area of the triangle is equal to the sum of the areas of the three subtriangles obtained by connecting z to the three vertices; then factor out the constant terms. This well-known independence is also used to represent the set of lotteries when there are three prizes.

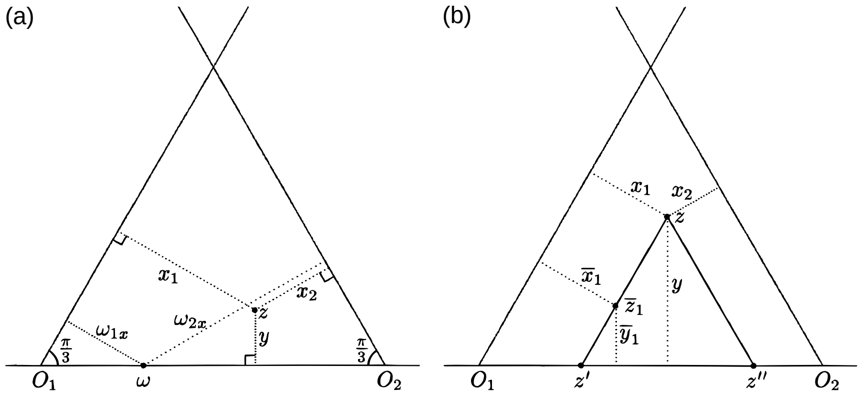

Figure 1: (a) **The points of the triangle are in one-to-one correspondence with the feasible allocations.** The two agents' consumptions of the private good at $z = (x_1, x_2, y)$ are measured by the distances from z to the slanted sides. The consumption of the public good is measured by the distance from z to the base of the triangle. (b) **When excludability is possible.** Starting from some allocation z, let z' denote the allocation given by the intersection of the base of the triangle with the line through z parallel to its left side, and let z'' be symmetrically defined. If exclusion is possible, agent 1 can be assigned any bundle on the segment from $z_1 = (x_1, y)$ to $z_1' = (x_1', 0)$, and agent 2 any bundle on the segment from $z_2 = (x_2, y)$ to $z_2'' = (x_2'', 0)$. The pair of bundles $\bar{z}_1 = (\bar{x}_1, \bar{y}_1)$ and z_2, where $\bar{y}_1 < y$, illustrates the case when agent 1 (and only agent 1) would be prevented from consuming the public good at the level at which it is produced.

images of two tangent curves are two tangent curves.[3] Figure 3 gives important examples of preference maps, before and after the transformation.

We now bring the two agents together. Let $R = (R_1, R_2)$ denote the profile of their preferences and $\omega_x = (\omega_{1x}, \omega_{2x})$ the profile of their endowments. An economy is a pair (R, ω_x). Starting from Figures 4a$_1$ and 4a$_2$, which contain all the relevant information concerning agents 1 and 2 respectively, namely their preferences and endowments, agent 2's map being drawn so as to face to the left, we perform the transformation just described, to obtain Figures 4b$_1$ and 4b$_2$. Then we slide these figures toward each other until ω_1 and ω_2 coincide. The equilateral triangle that results has the correct height $\omega_{1x} + \omega_{2x}$ to represent the set of allocations. It is Figure 4c. The point of coincidence is the image of the initial allocation ω. The top vertex of the triangle corresponds to devoting the entire

[3]On the other hand, equalities of angles are not generally preserved. Also, the images of two perpendicular lines are not perpendicular lines.

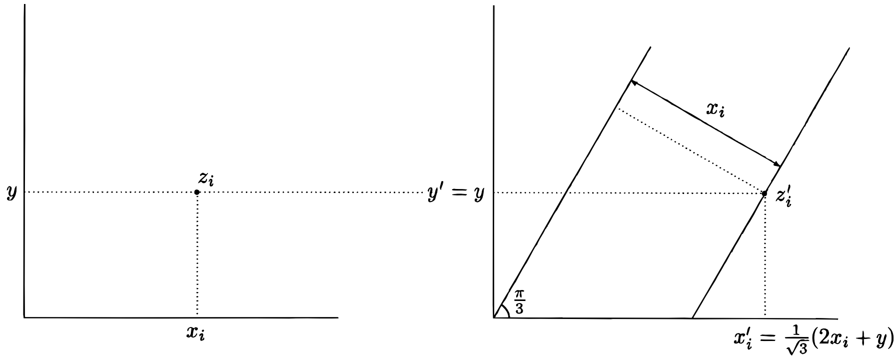

Figure 2: **Passing from the rectangular coordinates to the oblique coordinates.**

aggregate endowment of the private good to the production of the public good; it is the image of the allocation $(0,0,\omega_{1x} + \omega_{2x})$.[4]

3. Efficiency

An allocation is (Pareto)-**efficient for $(\boldsymbol{R},\boldsymbol{\omega_x})$** if there is no other allocation that all agents find at least as desirable, and at least one agent prefers. We denote by $P(R,\omega_x)$ the set of efficient allocations, or **Pareto set**, of (R,ω_x): Formally, $z \in P(R,\omega_x)$ if $z \in Z$ and there is no $z' \in Z$ such that for all $i \in N$, $z'_i\ R_i\ z_i$, and for at least one $i \in N$, $z'_i\ P_i\ z_i$. An allocation z is **weakly efficient for $(\boldsymbol{R},\boldsymbol{\omega_x})$** if there is no allocation z' that all agents prefer, that is, such that for all $i \in N$, $z'_i P_i z_i$. Graphically, the efficiency of an allocation is determined in the same way as in an Edgeworth box, that is, by checking separation of upper-contour sets, or tangency of indifference curves if preferences are smooth and the allocation is interior.

Figure 5b represents an economy in which agent 1's indifference curves are transversal to the public good axis. Agent 2's preferences are maximized on the feasible set along a nondegenerate interval of points in the left side of the triangle, and the top endpoint of that interval Pareto-dominates the others. (There may be other efficient points on that side, however, located above it, such as z'' in the figure.) Therefore, and even if preferences are strictly monotone, an allocation may be weakly efficient without being efficient.[5] [This is in contrast with the private good case.]

[4] I strongly recommend extending the axes and a few indifference curves beyond the triangle, as such extensions are of great help in remembering to consider boundary issues.

[5] This observation was made by Tian (1988) by means of a three-agent example, an example that is also discussed by Diamantaras and Wilkie (1996). However, the fact is also true in the two-person case, and Figure 5b shows that the Kolm triangle permits a very simple illustration of it.

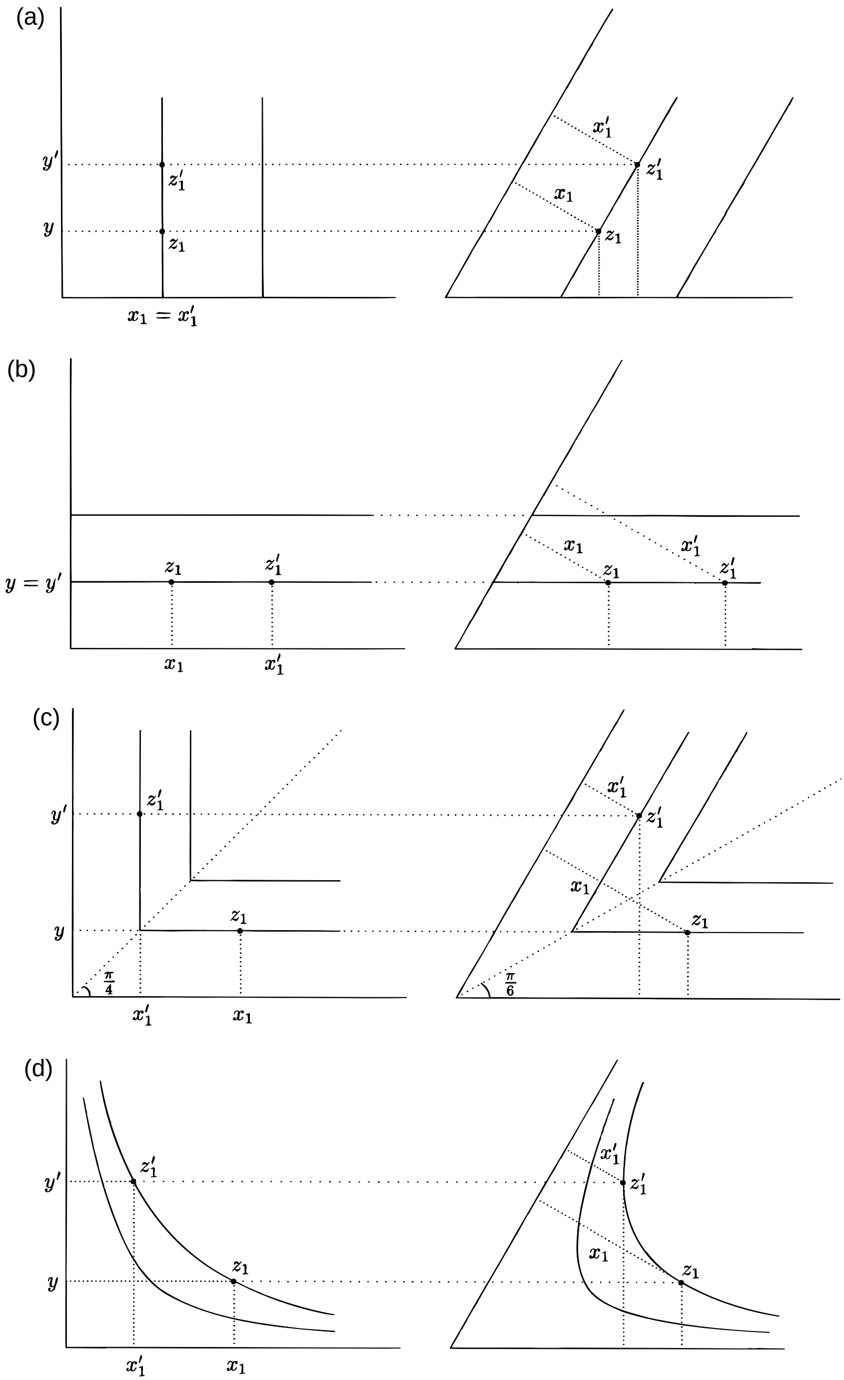

Figure 3: *Figure caption on facing page*

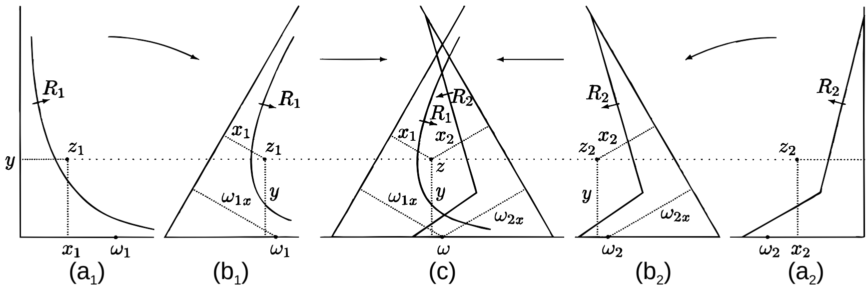

Figure 4: **Constructing the Kolm triangle.** The two preference maps, represented in panels (a$_1$) and (a$_2$), are subjected to the transformation illustrated in Figure 3, agent 2's map being drawn so as to face to the left. The transformed maps are given in panels (b$_1$) and (b$_2$). They are then pushed toward each other until the endowment points meet. The triangle that results, panel (c), has the correct height to represent the feasible set. The point *z* is a typical allocation.

Figures 6 and 7 illustrate other possibilities for the shape of the Pareto set, both in the Kolm triangle, but also in "welfare space," using numerical representations of preferences u_1 and u_2 chosen so that for all $i \in N$, $u_i(0) = 0$. In Section 7 we give additional examples. In most of our exposition, we limit ourselves to economies whose Pareto sets have no thick region. This property holds if preferences are strictly convex in $\mathbb{R}^2_{++}$.

Figure 6 depicts an economy in which no indifference curve passing through a positive point meets the axes (then, under our maintained assumptions on preferences the lowest indifference curve of each agent consists of the union of the nonnegative parts of the axes). If, furthermore, preferences are strictly convex in $\mathbb{R}^2_{++}$ (as is the case for, say, Cobb–Douglas preferences), the Pareto set is a curvilinear segment that meets each slanted side at only one point. These points are the unique maximizers of the agents' preferences over the feasible set.

Let us assume that each agent's preferences are strictly convex in $\mathbb{R}^2_{++}$. If, for example, agent 2's indifference curve through his component of agent 1's most preferred allocation is transversal to the right side, the

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Figure 3: **Examples of preference maps, before and after the transformation needed to use the Kolm triangle.** In each case, we indicate a pair of points that are indifferent to each other, before and after the transformation. (a) Linear preferences when the agent cares only about the private good. (b) Linear preferences when the agent cares only about the public good. (c) Leontieff preferences. (d) Cobb–Douglas preferences.

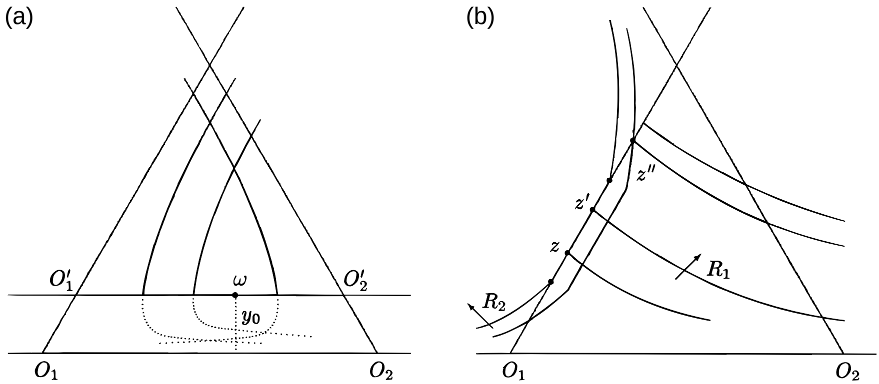

Figure 5: (a) **Accommodating a positive initial level of the public good.** If the initial level of the public good is positive, as it is at ω, and the good cannot be "built down," the feasible set is that part of the triangle that lies above the horizontal line through the endowment allocation. Even if indifference curves are asymptotic to the horizontal axis in the original consumption spaces, their restrictions to the feasible set, which is the smaller triangle with origins O_1' and O_2', are transversal to its base. (b) **Efficiency versus weak efficiency.** In this economy in which preferences are strictly monotone, an allocation such as z is weakly efficient but not efficient, being Pareto dominated by say, z'.

Pareto set includes a nondegenerate segment contained in that side (Figure 7).[6] Each slanted side of the triangle may in fact contain several intervals of efficient allocations. Note the lack of comprehensiveness[7] of the feasible set in welfare space. [Comprehensiveness holds in the private good case if goods are freely disposable.]

We should be careful in interpreting the tangency of indifference curves in a Kolm triangle: it means that the sum of the agents' marginal benefits at their respective bundles from a small increase in the public good level is equal to the marginal cost of that increase at the corresponding production point. [In an Edgeworth box, it means equality of the agents' marginal rates of substitution at their respective bundles.]

To see this, consider the smooth economy[8] of Figure 8c. Let $z = (x_1, x_2, y)$ be an interior efficient allocation. Draw through z the common tangency line to the agents' indifference curves. This line intersects the horizontal axis at the point $z' = (x_1', x_2', 0)$.

[6]Saijo (1990) illustrates the possibility.

[7]A subset of R' is "comprehensive" if whenever a point x belongs to it, any point y such that $x \geq y$ also belongs to it.

[8]An economy is "smooth" if at any interior allocation each agent's indifference curve through his component of the allocation has a unique line of support.

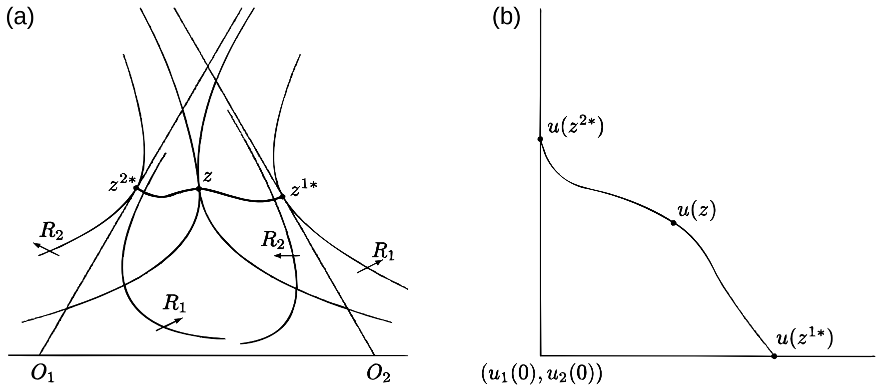

Figure 6: **The Pareto set when indifference curves passing through positive points do not meet the axes.** (a) It is a curvilinear segment connecting the two slanted sides of the triangle. (b) Its image in welfare space, using representations u_1 and u_2 such that for all $i \in N$, $u_i(0) = 0$.

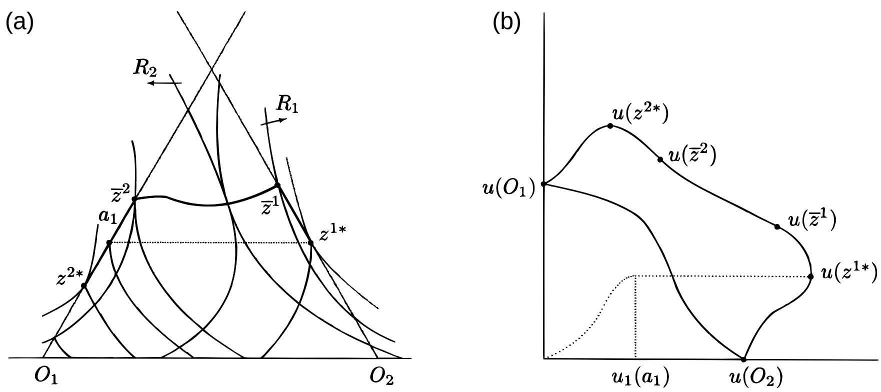

Figure 7: **The Pareto set when indifference curves are transversal to the axes.** (a) The Pareto set may contain nondegenerate intervals in the slanted sides of the triangle, here the intervals $[z^{1*}, \bar{z}^1]$ and $[z^{2*}, \bar{z}^2]$. In order to decrease agent 1's welfare from what it is at $z^{1*} = (x_1^{1*}, y^{1*})$, we make him dispose of his holdings x_1^{1*} of the private good, which brings him to the point $a_1 = (0, y^{1*})$. Any further decrease in his welfare requires decreasing the public good level, which would affect agent 2 negatively as well (unless the public good is excludable). (b) In welfare space, the image of the triangle is not comprehensive. Comprehensiveness is not recovered if the private good is freely disposable. (However, if the public good also is excludable then the property holds.)

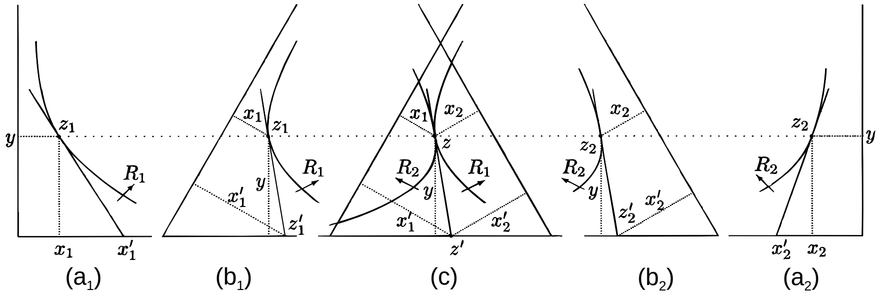

Figure 8: **Characterization of the efficient allocations.** Given an efficient allocation z, we identify the common line of support to the two indifference curves through z, draw the two images of this line in the rectangular axes, and calculate the slopes of these images.

We obtain agent 1's marginal benefit from a small increase in the public good level at $z_1 = (x_1, y)$ (his marginal rate of substitution there) by taking the absolute value of the inverse of the slope of the line connecting z_1 to $z_1' = (x_1', 0)$ in Figure 8a$_1$. This yields

$$MRS_1(z_1) = \frac{x_1' - x_1}{y}.$$

A similar calculation gives

$$MRS_2(z_2) = \frac{x_2' - x_2}{y}.$$

Therefore,

$$MRS_1(z_1) + MRS_2(z_2) = \frac{x_1' - x_1 + x_2' - x_2}{y} = \frac{x_1' + x_2' - (x_1 + x_2)}{y},$$

and since $x_1 + x_2 + y = x_1' + x_2' = \omega_{1x} + \omega_{2x}$, we have

$$MRS_1(z_1) + MRS_2(z_2) = 1.$$

The interpretation of this formula is simple. To determine the efficiency of an allocation, we compare the social marginal benefit of an increase in the public good level to its marginal cost. Because all agents can simultaneously consume the public good, this social marginal benefit is the *sum* of the agents' individual marginal benefits, and because the technology is linear with a one-to-one rate of transformation between input and output, its marginal cost is 1. At an efficient allocation, we should have equality. The marginal conditions for arbitrary numbers of private and public goods and arbitrary technologies are obtained in the standard way by maximizing the welfare of an arbitrary agent subject to achieving fixed welfare levels for the others and satisfying the technological constraints (Samuelson 1955).

A boundary case is illustrated in Figure 7. At a point such as z^{2*}, which is efficient, the sum of the agents' marginal benefits is greater than 1 (for agent 1, we use the limit of his marginal benefit along any sequence of interior bundles approaching z_1^{2*}). Conditions for an allocation, interior or not, to be efficient are derived by Campbell and Truchon (1988) in the one-private-good case (see Saijo 1990 for further details). Conley and Diamantaras (1996) present conditions for the case of an arbitrary number of private and public goods when preferences are continuous, convex, and locally nonsatiated but nonnecessarily smooth.

4. Lower and Upper Bounds on Welfares

A **correspondence** defined on some domain of economies is a mapping that associates with each economy in the domain a nonempty set of allocations. An example is the correspondence that associates with each economy its set of efficient allocations, the **Pareto correspondence**. In the remaining sections we look for correspondences satisfying certain criteria of desirability. The letter φ is our generic notation for correspondences. Given $\mathcal{R}$, a class of preference relations, we denote by $\mathcal{R}^N$ the cross-product of two copies of $\mathcal{R}$ indexed by the members of N. We slightly abuse notation and also use the N superscript to denote a class of economies involving the group N. For instance, $\mathcal{E}^N$ is our notation for the product $\mathcal{R}^N \times \mathbb{R}^N_+$.

We begin with "welfare bounds." The first requirement is simply that each agent should find his assigned bundle at least as desirable as his endowment. The correspondence that associates with each economy the set of allocations meeting this requirement is the **endowments lower bound correspondence**: $\underline{B}_{end}(R,\omega_x) = \{z \in Z\colon \text{for all } i \in N,\ z_i\ R_i\ \omega_i\}$. We refer to its intersection with the Pareto correspondence as the endowment's lower bound *and* Pareto correspondence, a correspondence that we denote by $\underline{B}_{end}P$. We use the same language and notational conventions in the rest of this essay. In Figure 9a, the image of this intersection is the broken segment connecting a, $\bar{z}^2$, and b, and in Figure 9b it is the curvilinear segment from a to b. The endowments lower bound is often called "individual rationality." It can indeed be understood as a participation constraint when agents have the right to dispose of their endowments as they wish.

A stricter requirement is that each agent should find his assigned bundle at least as desirable as any one of the bundles he can reach from

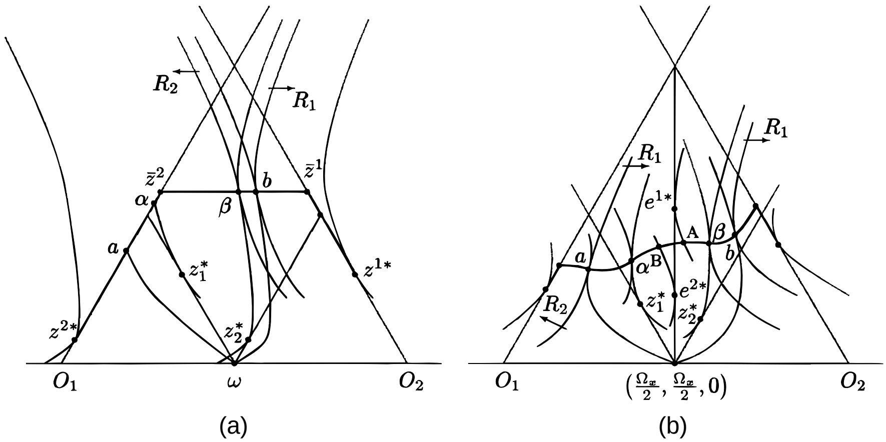

Figure 9: **Bounds on welfares.** (a) In this quasi-linear example, the Pareto set is the broken segment connecting z^{2*}, $\bar{z}^2$, $\bar{z}^1$, and z^{1*}. The efficient allocations such that each agent i finds his assigned bundle at least as desirable as his endowment are the points of the broken segment connecting a, $\bar{z}^2$, and b. The efficient allocations such that each agent finds his assigned bundle at least as desirable as the bundle he prefers among those he could reach from his endowment if given unhampered access to the technology are the broken segment connecting α, $\bar{z}^2$, and β. (b) If agents are endowed with an equal share of the social endowment, the previous notions can be adapted by simply setting for each $i \in N$, $\omega_{ix} = \Omega_x/2$. Another requirement is that each agent should find his assigned bundle at most as desirable as the equal-bundle allocation he prefers. Agents 1 and 2's preferred equal-bundle allocations are e^{1*} and e^{2*}, respectively, and the set of efficient allocations satisfying this requirement is the curvilinear segment from A to B.

his endowment if given unhampered access to the technology, but assuming the other agent contributes nothing to public good production. The correspondence that associates with each economy its set of allocations meeting this requirement is the (technology)-**augmented-endowments lower bound correspondence**: $\underline{B}_{augend}(R, \omega_x) = \{z \in Z\colon$ for all $i \in N$ and all $z_i' = (x_i', y') \in \mathbb{R}^2_+$ such that $x_i' + y' = \omega_{ix}$, $z_i R_i z_i'\}$. Illustrations of its intersection with the Pareto correspondence are the broken segment connecting α, $\bar{z}^2$, and β in Figure 9a and the curvilinear segment from α to β of Figure 9b.

In the n-person case, when the requirement is imposed on an allocation that there should be no group of agents and no list of bundles that they can reach from their endowments if given unhampered access to the technology, and again assuming that the complementary group contributes nothing to public good production, and that each of them finds the allocation at least as desirable and one of them prefers it, we obtain the notion of core proposed by Foley (1967). Formally, $C(R,\omega_x) = \{z \in Z$: there is no $S \subseteq N$ and $((x'_i)_{i \in S}, y') \in \mathbb{R}^S_+ \times \mathbb{R}_+$ such that (i) $\sum_S x'_i + y' = \sum_S \omega_{ix}$, and (ii) for all $i \in S$, $z_i R_i (x'_i, y')$, and for some $i \in S$, $z_i P_i (x'_i, y')\}$.

In certain situations, it is natural to model an economy as a list of preference relations together with a **social endowment** of the private good. Denoting such an endowment by $\Omega_x \in \mathbb{R}_+$, an economy is then a pair $(R,\Omega_x) \in \mathcal{R}^N \times \mathbb{R}_+$. We denote a generic domain of such economies by $\mathcal{F}^N$. Appealing requirements for this version of the model are obtained by first dividing the social endowment equally and then applying the ideas formulated in the previous paragraphs. We speak then of an allocation meeting the **equal-division lower bound**, or meeting the **augmented-equal-division lower bound**, or belonging to the **equal-division core**.

The bound defined next is obtained by identifying first for each agent the allocation(s) he prefers *among the allocations at which the other agent consumes an equal bundle*. We demand of an allocation that each agent should find it *at most as desirable* as the equal-bundle allocation(s) he prefers. In Figure 9b the set of equal-bundle allocations is the vertical segment containing the top vertex of the triangle and for each $i \in N$, maximizing R_i on this segment gives the point denoted by e^{i*}. The curvilinear segment from A to B is the set of efficient allocations satisfying the upper bounds just introduced. We denote the correspondence defined in this way, which we name the **preferred-equal-bundle-allocation upper bound correspondence**, by $\bar{B}_{peba}$: $\bar{B}_{peba}(R,\Omega_x) = \{z \in Z$: for all $i \in N$, $e^{i*}_i R_i z_i$, where $(e^{i*}_i, e^{i*}_i) \in Z$ and for all $z' \in Z$ with $z'_i = z'_j$ we have $e^{i*}_i R_i z'_i\}$.

We will sketch a proof that in the two-person case there are efficient allocations meeting both the augmented-equal-division lower bound and the preferred-equal-bundle-allocation upper bound by exhibiting an allocation rule having all the properties. It is the rule that selects the efficient allocation(s) at which each agent is indifferent between his assigned bundle and the best bundle he could reach from an equal share of the social endowment if given unhampered access to a reference technology that is λ times as productive as the actual technology; for each level of the input, the amount of output produced is λ times what it is in the actual technology. If Y designates the actual technology, let Y^λ denote this adjusted technology. Formally, we are defining the set $\{z \in P(R,\Omega_x)$: there is $\lambda \in \mathbb{R}_+$ such that for all $i \in N$, $z_i \, I_i \, z^*_i$, where $z^*_i \in Y^\lambda + \{(\Omega_x/2, 0)\}$ and for all $z'_i \in Y^\lambda + \{(\Omega_x/2,0)\}$, $z^*_i \, R_i \, z'_i\}$.

In Figure 10a the opportunity sets faced by the two agents when the technology is subjected to this adjustment are lines that are symmetric with respect to the vertical segment containing the top vertex and that originate from the allocation $(\Omega_x/2, \Omega_x/2, 0)$. For λ small, the lines are very flat, and as λ increases, they rotate around the endowment point. Let us now take, for each λ, the image in welfare space of the pair of maxi-

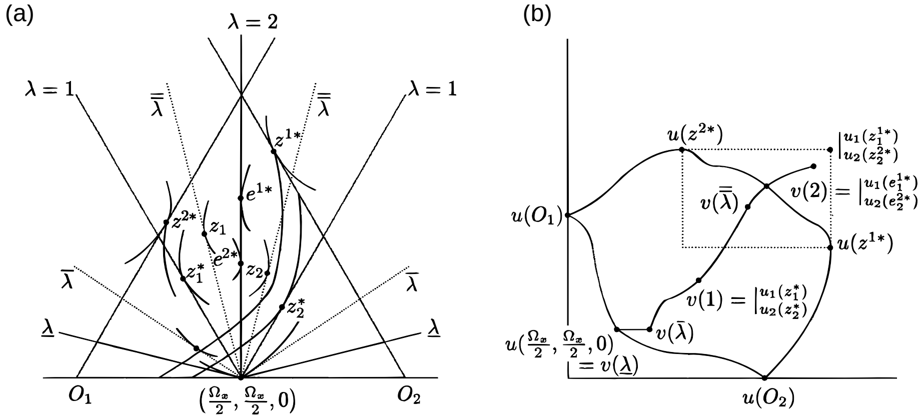

Figure 10: **Compatibility of the augmented-equal-division lower bound and the preferred-equal-bundle-allocation upper bound.** (a) For each value of the parameter λ, we maximize the two agents' preferences on an opportunity set defined by giving them access to the technology Y^{λ} and an endowment of the private good equal to $\Omega_x/2$. (b) The image in welfare space of the pair of maximizing bundles, as λ varies, is a monotone path that has to intersect the Pareto set. The inverse image of this point in commodity space is an efficient allocation meeting the two bounds.

mizers. The point so obtained, designated as $v(\lambda)$, describes a monotone path that emanates from the downward-sloping curvilinear segment connecting $u(O_1)$ to $u(O_2)$ (Figure 10b). For $\lambda = 1$, we obtain the point $(u_1(z_1^*), u_2(z_2^*))$ and for $\lambda = 2$, the point $(u_1(e_1^{1*}), u_2(e_2^{2*}))$. The latter is outside of the feasible set and below $(u_1(z_1^{1*}), u_2(z_2^{2*}))$. These conclusions together imply that the path crosses the Pareto set.

Note that it is because the image in welfare space of the feasible set is not necessarily comprehensive that we have to prove that the intersection of the path with its boundary is indeed an undominated feasible point. When indifference curves are asymptotic to the axes, this conclusion holds automatically.

The preferred-equal-bundle-allocation upper bounds are equivalently obtained by imagining, for each agent, an economy in which the other agent has preferences identical to his, and calculating the common welfare level that he and his clone would enjoy under efficiency and the requirement, whose formal statement follows, that two agents with the same preferences be treated identically in terms of welfare. Therefore, the preferred-equal-bundle-allocation upper bound coincides with what can be called the **identical-preferences upper bound** (Moulin 1990).

Equal treatment of equals: For all $(R,\Omega_x) \in \mathcal{F}^N$, all $z \in \varphi(R,\Omega_x)$, and all $i,j \in N$, if $R_i = R_j$, then $z_i I_i z_j$.

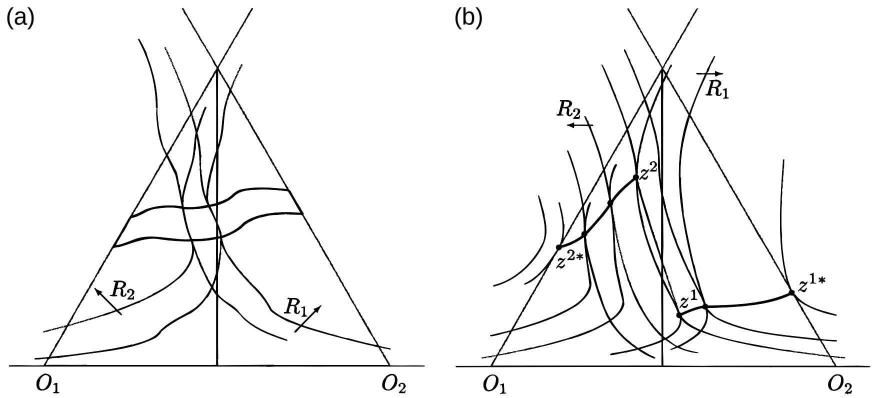

Figure 11: **Equal-cost-shares.** (a) An efficient equal-cost-share allocation is any point of intersection of the Pareto set with the vertical segment containing the top vertex. Here, the Pareto set is a two-dimensional area that intersects this vertical segment. (b) When preferences are not convex there may be no efficient and equal-cost-share allocation.

Another requirement is that both agents should contribute the same amount of the private good to the production of the public good. The allocations at which this is achieved consist of the vertical segment containing the top vertex of the triangle. The correspondence that associates with each economy these allocations is the **equal-cost-share** correspondence. Now, say that an allocation is **envy-free** if each agent finds his assigned bundle at least as desirable as the bundle assigned to the other agent, and let F denote the correspondence that associates with each economy its set of envy-free allocations: $z \in F(R, \Omega_x)$ if $z \in Z$ and for all $i, j \in N$, $z_i R_i z_j$. It is clear that this correspondence coincides with the equal-cost-share correspondence.

Asking whether the equal-cost-share and Pareto correspondence is nonempty is asking whether there are efficient allocations on the vertical segment containing the top vertex. The answer is yes if the Pareto set is "in one piece." Otherwise it may be no, as illustrated in Figure 11b.[9] The most general existence result is due to Diamantaras (1992). Its central assumption is that for each efficient allocation, the set of Pareto-indifferent allocations is contractible.

5. The Lindahl correspondence

The correspondence proposed by Lindahl (1919) is an important theoretical tool for allocating resources in economies with public goods. In many

[9]This example is adapted from a similar example given by Varian (1974) to show that envy-free and efficient allocations may not exist in a private good economy with nonconvex preferences.

ways, it is a counterpart of the Walrasian correspondence. It too requires each agent to maximize his preferences subject to a budget constraint. The difference is that here each agent faces an *individualized* price for each public good.

It may be useful to introduce the Lindahl correspondence by first explaining in the Kolm triangle what would go wrong if we attempted to use the Walrasian correspondence itself and charged the same prices to both agents. At equilibrium, they should of course demand the same quantity of the public good, and the problem is that equilibria might not exist. Indeed, if the common price can be chosen so that the maximizing bundles have the same public good component, too much (Figure12a) or too little (Figure 12b) of the private good may be collected to produce the public good at that level. For aggregate feasibility to be satisfied as an equality, the two budget lines should coincide. This can happen only if this common budget line is vertical, but then the public good components of the maximizing bundles will typically differ (Figure 12c).

On the other hand, if a single line from the initial allocation is used to define the two budget sets and this line is allowed to differ from the vertical—meaning that the two agents are charged different prices—feasibility becomes compatible with existence. This observation is the basis for the definition of the Lindahl correspondence.

We begin with a discussion of this important definition in a general situation when there are any number of private and public goods and any number of producers. In that case we face each consumer with his own price for each of the public goods, keeping the prices of the private goods equal across agents. Producers maximize profits in the usual way except that the price they face for each public good is the sum of the individu-

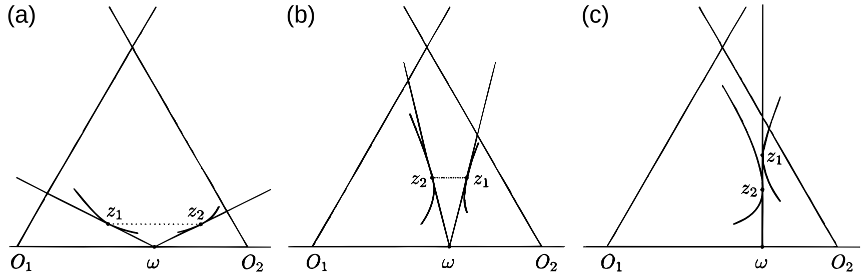

Figure 12: **What is wrong with operating the Walrasian correspondence?** In each panel, both agents face the same prices. (a) At these common prices, they happen to demand the public good at the same level, but the sum collected from them to produce it at that level is greater than necessary to cover its production cost. (b) Here, the sum collected is too small. (c) Here, the two agents do not agree on the level at which to build the public good.

alized prices paid by the consumers for that good. We endow consumers with fractional ownerships of the firms. For simplicity, let us now assume that there is only one private good and one firm. Let $\theta = (\theta_i)_{i \in N} \in \Delta^N$ be the list of shares according to which its profits are distributed to consumers.[10] The production set Y is taken from some admissible class $\mathcal{Y}$. Altogether, and denoting the list $(\omega_{1x}, \ldots, \omega_{nx})$ by ω_x, an economy is a quadruple $(R, \omega_x, \theta, Y) \in \mathcal{R}^N \times \mathbb{R}^N_+ \times \Delta^N \times \mathcal{Y}$. Instead of choosing the price vector faced by agent i in the unit simplex, let us impose the normalization condition that its first coordinate has value 1 (in the rest of this essay, we will limit our attention to economies for which this normalization condition is justified). Then $(1, \pi_i)$ is the price vector faced by agent i, π_i having as many coordinates as there are public goods. In the following definition, we assume that there is only one public good.

Definition 1: *Given $e = (R, \omega_x, \theta, Y) \in \mathcal{R}^N \times \mathbb{R}^N_+ \times \Delta^N \times \mathcal{Y}$, the allocation* z **Z is a Lindahl allocation for e** *if there is a list $(\pi_i)_{i \in N} \in \mathbb{R}^N_+$ such that for each $i \in N$, z_i maximizes R_i on the budget set $\{z'_i = (x'_i, y') \in \mathbb{R}^2_+ : x'_i + \pi_i y' \leq \omega_{ix} + \theta_i A$, where $A \in \mathbb{R}$ is the maximum of $x'' + (\sum \pi_i) y''$ for $(x'', y'') \in Y\}$.*

Let L designate the correspondence that associates with each economy its set of Lindahl allocations. A counterpart of the **first welfare theorem holds**: under standard assumptions on preferences, a Lindahl allocation is efficient.

Since the triangle pertains to the case of a linear technology, at equilibrium the profits can only be zero, and the shares are immaterial. When the individual prices a consumer faces vary, we trace his **demand function** (or correspondence). By analogy with the private good case, we call the loci of maximizers the agent's **offer surface**, or **offer curve** if there are only two goods. We denote agent i's offer curve by h_i. In the Kolm triangle, we obtain the Lindahl allocations as the points of intersection of these loci that belong to the cone defined by the lines parallel to the slanted sides emanating from the initial allocation, this allocation being in general excluded. The allocation z of Figure 13c is an example. The line passing through ω and z defines two budget sets, one for each agent. Then, given our normalization condition, we have $\pi_1 + \pi_2 = 1$, as we saw earlier when we discussed the marginal conditions for efficiency. In the figure, $\pi_1 > \pi_2$.

Figure 14 gives an example of an economy with quasi-linear but nonconvex preferences in which Lindahl allocations do not exist.

To further illustrate the notion, let us calculate the Lindahl allocation (it is unique) of a Cobb–Douglas economy (Figure 15a). In that case, the Pareto set is a linear segment connecting the two agents' most preferred allocations. Agent 1's offer "curve" consists of a half-line parallel to the

[10] By Δ^N we mean the unit simplex in $\mathbb{R}^N$.

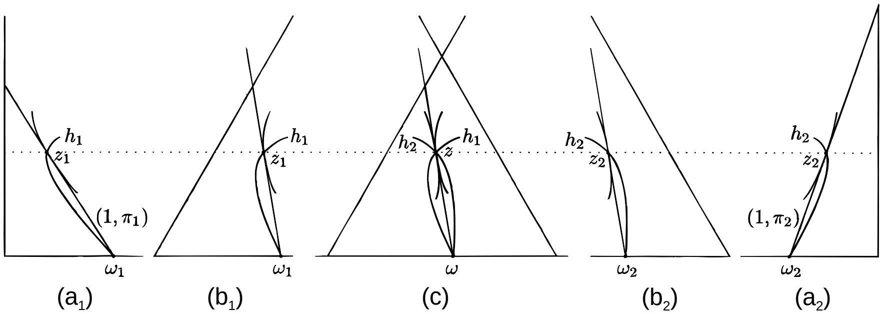

Figure 13: **The Lindahl correspondence.** Each agent $i \in N$ faces his own price vector, denoted by $(1, \pi_i)$. If π_1 and π_2 are chosen so that $\pi_1 + \pi_2 = 1$, then in the triangle the corresponding budget lines coincide, as they do in panel (c), and if the two maximizers have the same ordinate, as happens there, they define a feasible allocation.

left side of the triangle together with the nonnegative part of the horizontal axis. Agent 2's offer curve is described in a symmetric way. The Lindahl allocation is given by the point of intersection of the two slanted half-lines.

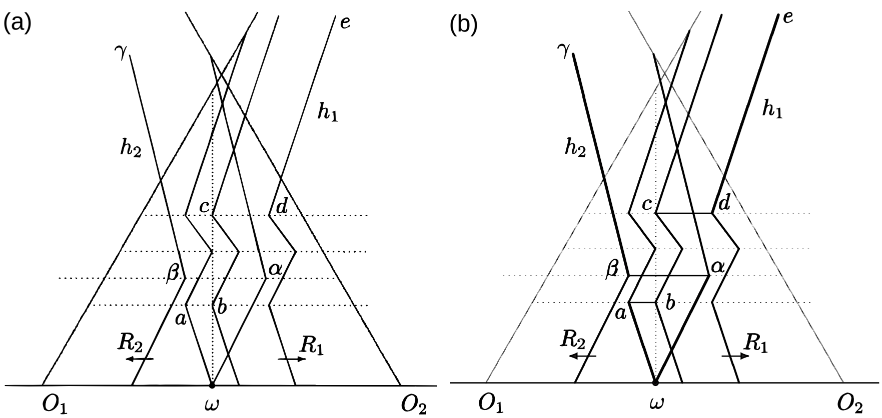

Figure 14: **Lindahl allocation may not exist in economies with nonconvex preferences.** (a) In this quasi-linear economy, agent 1 has nonconvex preferences. (b) Agent 1's offer curve is in two pieces, the broken segment connecting ω, a, and b, and the broken segment connecting c, d, and extending beyond e until it reaches the left slanted side, then following that side upward. Agent 2's offer curve is the broken segment connecting ω, α, β, and extending beyond γ until it reaches the right slanted side, then following it upward; it passes through the gap in agent 1's offer curve. As a result, there is no Lindahl allocation.

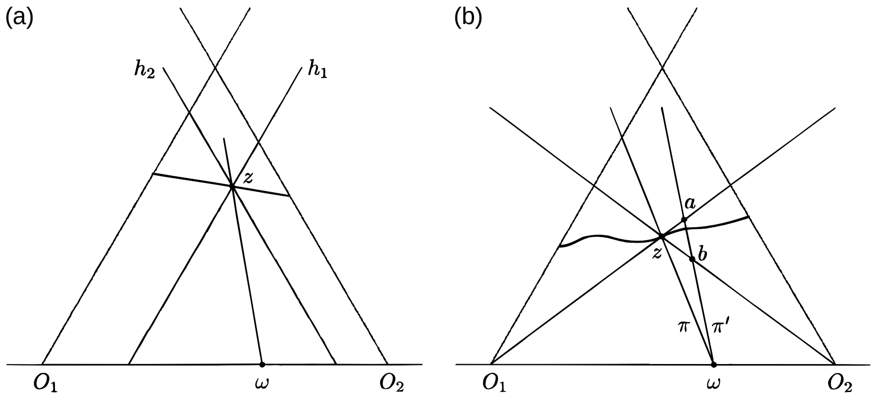

Figure 15: **The Lindahl allocation when preferences are homothetic.** (a) The Lindahl allocation of a Cobb–Douglas economy. Agent 1's offer "curve" h_1 consists of a half-line parallel to the left side of the triangle together with the nonnegative part of his horizontal axis. Agent 2's offer curve h_2 is defined symmetrically. The offer curves intersect at a unique point that qualifies as a Lindahl allocation, the point z. (b) The Lindahl allocation for an economy with preferences that are strictly convex in $\mathbb{R}^2_{++}$ and homothetic is unique: If z is such an allocation, with supporting prices π, at no other prices can the agents' two maximizers coincide. At the prices π', for instance, agent 1's maximizer is above a and agent 2's maximizer is below b.

The standard **existence proof for Lindahl allocations** (Foley 1967; Milleron 1972) consists of augmenting the commodity space by (i) treating the public good as consumed by each agent as a separate private good, (ii) correspondingly extending the endowment and the preferences of each agent to the augmented space (note that according to his new preferences, an agent is indifferent between any two allocations differing only in the consumptions of the artificial private goods associated with the other agents), (iii) extending the production set in such a way that the production of y units of the public good corresponds to the joint production of y units of each of the artificial private goods, (iv) proving existence of a Walrasian allocation in that artificial economy.

For a homothetic economy, there is a unique (up to Pareto-indifference) Lindahl allocation, as illustrated in Figure 15b. This uniqueness result is obtained by further exploiting the technique just outlined to prove existence. First note that if preferences are homothetic in the original space, their extensions to the extended space still are. In the extended space, endowments are still equal so that no matter what prices are quoted, incomes remain in the same proportions. Homotheticity of preferences and an income distribution that is independent of prices imply that aggregate demand satisfies the weak axiom of revealed preference, which in turn implies uniqueness of equilibrium (Mas-Colell, Whinston, and Green 1995).

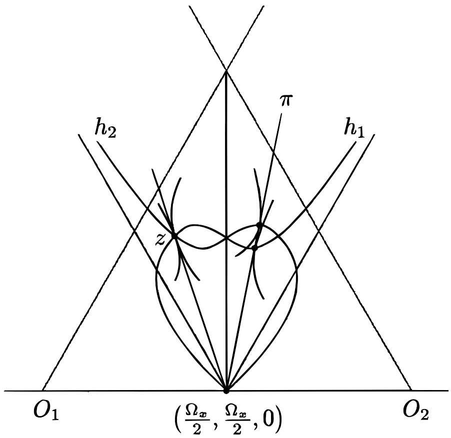

Figure 16: **The Lindahl correspondence does not necessarily treat identical agents identically.** At the Lindahl allocation *z*, the two agents, who have identical offer curves, do not receive identical bundles. Also, starting from the disequilibrium prices π, at which agent 1's demand for the public good is smaller than agent 2's, the natural adjustment consisting of decreasing the price of the public good faced by agent 1 and correspondingly increasing the price faced by agent 2 leads us away from the symmetric Lindahl allocation. Of the three Lindahl allocations, only the two asymmetric ones are locally stable under this adjusment process.

A counterpart of the **second welfare theorem** also holds under standard assumptions: Given some efficient allocation, a distribution of the aggregate endowment of the private good can be found such that the allocation is a Lindahl allocation for the economy with individual endowments so defined. Indeed, there is a line separating the upper contour sets there whose slope has to be intermediate between the slopes of the two slanted sides, so that it will cross the base of the triangle at a point that can be taken as the desired initial allocation (Conley and Diamantaras 1996). We used this fact in deriving the marginal conditions for efficiency (cf. Figure 8).

On the other hand, the Lindahl correspondence does not satisfy *equal treatment of equals* (Figure 16). [This is in contrast to the Walrasian correspondence.] Of course, the *set* of Lindahl allocations for each economy does respect such symmetries but the correspondence of the symmetric Lindahl allocation is not *upper-semicontinuous with respect to preferences* (Champsaur 1976).

6. Common Mistakes

In this section, we list errors commonly made in the use of the Kolm triangle. Most of these errors are due to an insufficient understanding of

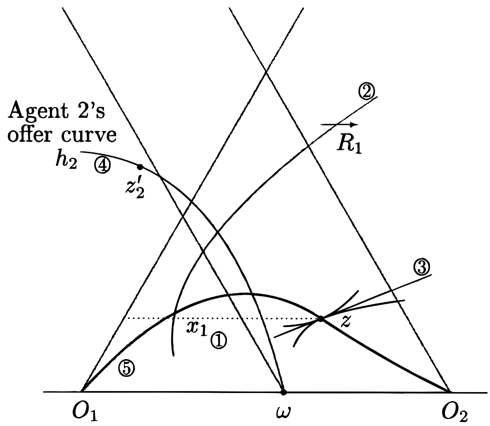

Figure 17: Five common mistakes.

how it differs from the Edgeworth box. They are illustrated in Figure 17 and numbered according to the list below:

Common mistake 1: Measuring the amount of the private good consumed by an agent at an allocation z by the horizontal distance from z to his slanted axis (instead of by the orthogonal distance from z to his slanted axis).

Common mistake 2: Violating monotonicity of preferences with respect to the public good. The top of agent 1's indifference curve should not be flatter than the left side of the triangle.

Common mistake 3: Giving a line of support to an allocation, here z, a slope that is not intermediate between the slopes of the two slanted sides of the triangle.

Common mistake 4: Drawing offer curves that bend down too much. The only way for a point such as z_2' to be part of agent 2's offer curve would be for him to face a negative price for the public good, but if he did, then under monotonicity of preferences, he would have no maximizer on his budget set. Therefore, the offer curve should lie above the line through his endowment parallel to the right side of the triangle.

Common mistake 5: Thinking of the Pareto set as necessarily being a curve connecting the two origins, a shape that is possible but rare.

7. More on the Pareto correspondence

In the remaining sections of this essay, we exploit the triangle construction to derive a number of interesting results. We start with several examples further illustrating the variety of shapes of the Pareto set.

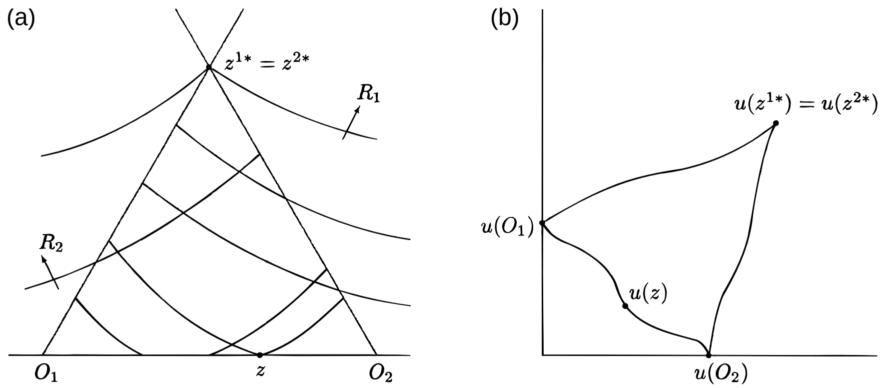

Figure 18: **The Pareto set in the extreme case when both agents have the same most preferred allocation.** (a) Here, each agent maximizes his preferences over the triangle at the point where the aggregate endowment of the private good is entirely devoted to the production of the public good. The allocation represented by the top vertex Pareto dominates every other allocation. (b) In welfare space, the corresponding vector of welfares dominates all other feasible vectors of welfares.

Figure 18 depicts an economy in which each agent's most preferred allocation is the allocation at which all of the private good available is devoted to the production of the public good. The Pareto set reduces to that allocation.

The Pareto set may be a curve connecting one of the origins to the top vertex of the triangle (Figure 19a). For instance, if agent 1's affinity for

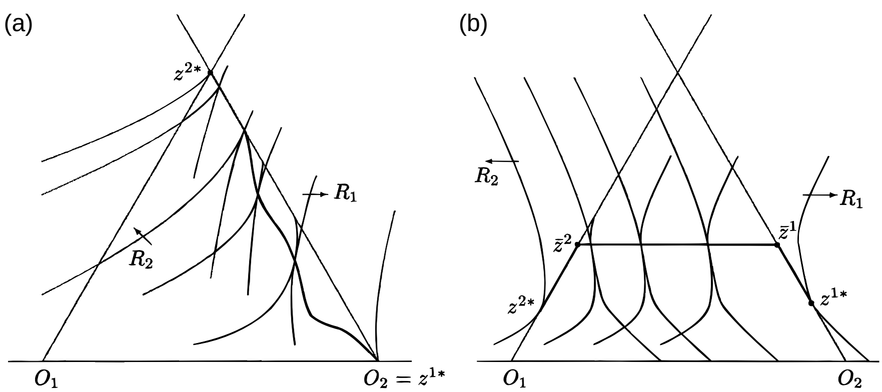

Figure 19: **Pareto sets for two other examples.** (a) Economy whose Pareto set is a curve joining O_2 to the top vertex of the triangle. (b) Pareto set for a quasilinear economy in which the consumption of the private good is required to be nonnegative. It consists of the segments $[z^{2*}, \bar{z}^2]$, $[\bar{z}^2, \bar{z}']$, and $[\bar{z}', z^{1*}]$.

the private good is so strong that alone he would choose to consume the entire social endowment of that good as such, his most preferred allocation is O_2; if agent 2's affinity for the public good is so strong that alone he would choose to devote the entire social endowment of the private good to producing it, his most preferred allocation is the top vertex of the triangle.[11]

We continue with two important special cases, quasi-linear economies and homothetic economies. In a quasi-linear economy, each agent's preference map can be obtained by translating any one of his indifference curves parallel to the private good axis by an arbitrary, positive or negative, amount.[12] When this invariance property is imposed, the nonnegativity requirement on the consumption of the private good is indeed usually dropped, as it makes the analysis particularly simple. Then the feasible set is the half-space above the horizontal axis; and if, additionally, preferences are convex, the Pareto set is a horizontal line or band. If the nonnegativity assumption is maintained, the term "quasilinear" refers to a map obtained by restricting to the nonnegative quadrant a map as just defined. Then, the Pareto set contains a horizontal segment or band connecting the two slanted sides of the triangle, as well as (usually) nondegenerate segments in the slanted sides (Figure 19b). If preferences are not convex, the Pareto set may be a union of bands when the nonnegativity constraint is not imposed, and otherwise it may also include several segments in each of the slanted sides of the triangle.

Conversely, assuming for simplicity that preferences are strictly convex, given any horizontal line and given any preference relation for agent 1 whose maximizer on the triangle is a point whose ordinate is no greater than the ordinate of that line, there is a preference relation for agent 2 such that the Pareto set of the economy so defined is the line. When the nonnegativity condition is imposed, given any set consisting of two segments lying in the slanted sides and of a horizontal segment connecting them at their top extremities, and given any preference relation for agent 1, whose maximizer on the triangle is the lowest point of the segment lying in the right side, there is a preference relation for agent 2 such that the Pareto set of the economy so defined is the set.

The Pareto set of a homothetic economy can be described in the simple case when preferences are strictly convex in $\mathbb{R}^2_{++}$, as follows. First, given an economy with preferences R that are strictly convex in $\mathbb{R}^2_{++}$ and homothetic, and given $z \in P(R, \omega_x)$ that is interior to the triangle, draw the two rays through z (Figure 20a). The rays divide the triangle into four regions. No point of the Pareto set can be located in the northern and southern regions (also excluded are the boundaries of these two regions). For instance, at a point such as $\tilde{z}$, any line of support to agent 1's upper-

[11]Topological properties of the Pareto set are discussed by Diamantaras and Wilkie (1996).

[12]In the slanted axes, this invariance property with respect to horizontal translations is preserved. So is homotheticity.

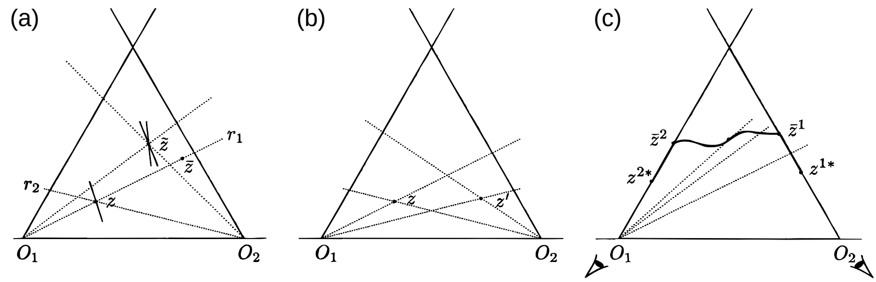

Figure 20: **The Pareto set of an economy with strictly convex and homothetic preferences is a doubly visible curve.** (a) If z is an interior efficient allocation, no point below both rays passing through z or above both rays, can also be official. At points such as $\tilde{z}$ and $\bar{z}$, the lines of support to the two agents' upper-contour sets necessarily cross. (b) If z' is efficient as well, then additional allocations are eliminated as possible efficient allocations. (c) Altogether, the Pareto set is a curve that is "visible" from both origins; an observer standing at O_1 or O_2 would see the whole of it. However, it may include two segments in the slanted sides of the triangle that are "barely" visible: the segments $[z^{2*}, \bar{z}^2]$ and $[z^{1*}, \bar{z}^1]$.

contour set is steeper than any line of support at z, whereas the opposite holds for agent 2. Since they have a common line of support at z, $\tilde{z}$ cannot be efficient. Repeating the argument for an efficient point in either the western or eastern regions, such as z' in Figure 20b, we obtain further constraints on where the Pareto set lies. The Pareto set could also contain segments in the slanted sides of the triangle, such as the segments $[z_1^*, \bar{z}_1]$ and $[z_2^*, \bar{z}_2]$. Altogether, we conclude that the set is "doubly visible" (Figure 20c): Imagine an observer standing at either origin and suppose that the Pareto set is opaque, then, the observer could still see the entire set from below.[13] (We include the limit case of segments contained in the slanted sides in our definition of visibility.)

Note that a homothetic preference relation can be described by means of the mapping that associates with each angle $\theta \in [0, \pi/2]$ the angles made with the horizontal axis by the lines of support to an upper-contour set at a point different from the origin where the curve crosses the ray, making an angle θ with the horizontal axis (Figure 21). If the relation is smooth, this mapping is a function on $]0, \pi/2[$; if in addition the relation is convex, the function is nowhere decreasing; if it is strictly convex in $\mathbb{R}^2_{++}$, the function is increasing. The same properties hold in the slanted axes, and we will denote by f_i the mapping with domain $[0, \pi/3]$ describing in these axes agent i's homothetic preference relation.

[13]Alternatively, we can say that the area below the Pareto set is "doubly star-shaped with centers O_1 and O_2."

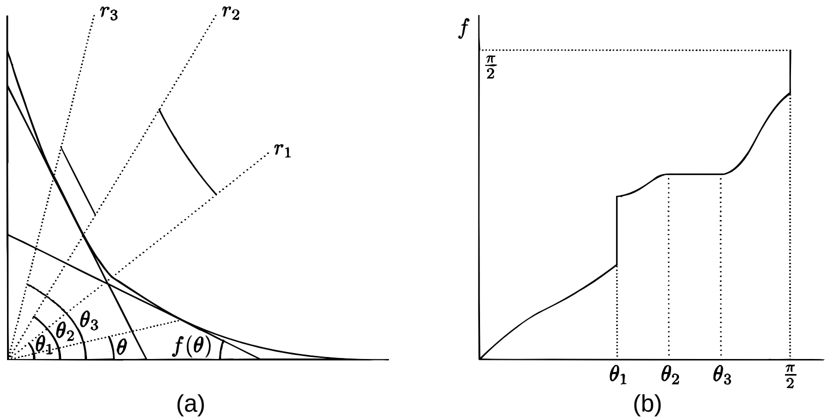

Figure 21: **Describing a homothetic preference relation.** (a) A homothetic preference relation can be described by means of the mapping f that associates with each ray through the origin, defined by its angle with the horizontal axis, the interval of angles made by lines of support to the upper-contour set at any point of the ray (except for the origin). (b) The graph of this mapping may have vertical segments (they correspond to rays along which the preference relation has kinks, such as the one defined by θ_1), and flat segments (they correspond to cones in which indifference curves are parallel linear segments, such as the cone defined by the segments of angles $[\theta_2, \theta_3]$).

Fix ω_x. We refer to as **regular** a preference relation that in addition to being continuous and monotone is strictly convex in $\mathbb{R}^2_{++}$ and smooth. For regular and homothetic preferences, it turns out that there is no property other than double visibility that the Pareto set has: Given any doubly visible curve C in a triangle of height $\sum \omega_{ix}$, there is a pair of regular and homothetic preference relations R_1 and R_2 such that C is the Pareto set of the economy (R_1, R_2, ω_x). In, fact, we can say more. Call a pair (C, R_1) of a doubly visible curve C and a regular and homothetic preference relation R_1 **compatible** if R_1 is maximized on the triangle at the lowest point of C on the right side. Then, given a compatible pair (C, R_1) we claim that there is a regular and homothetic preference relation R_2 for agent 2 such that C is the Pareto set of the economy (R_1, R_2, ω_x).[14]

To prove this claim, let (C, R_1) be a compatible pair, with R_1 described by the function f_1 in the slanted axes. Let Θ be the range of angles made by the rays from O_2 that intersect C (Figure 22a). Let z^{2*} and $\bar{z}^2$ be the lowest and highest points of C on the left side. Let θ^* and $\bar{\theta}$ be the angles defining the rays from O_2 passing through z^{2*} and $\bar{z}^2$. Given a ray from O_2 of angle $\theta \in [\bar{\theta}, \pi/3]$, let z be its point of intersection with C (Figure 22b).

[14]This is a counterpart of a result for private good economies described in Thomson (1998).

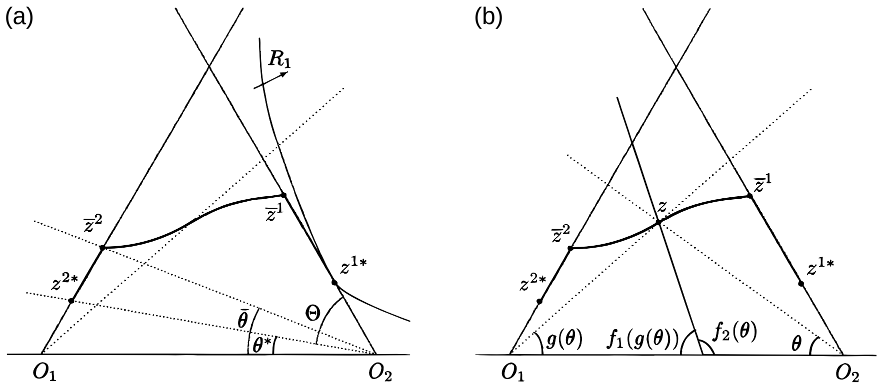

Figure 22: **Rationalizing a doubly visible curve as the Pareto set of a homothetic economy.** (a) Given a doubly visible curve C and a regular and homothetic preference relation R_1 for agent 1, if R_1 is maximized over the triangle at the lowest point of C on the right side (z^{1*}), there is a regular and homothetic preference relation R_2 for agent 2 such that C is the Pareto set of the economy (R_1, R_2). (b) Construction of the function f_2 defining R_2.

Let $g(\theta)$ be the angle made by the ray through O_1 passing through z. The line of support to agent 1's upper-contour set at z makes an angle $f_1(g(\theta))$ with the horizontal axis. For z to be efficient, we have to choose $f_2(\theta)$ equal to $\pi - f_1(g(\theta))$. Since f_1 is increasing and g is decreasing, f_2 is increasing, as required. Since z^{2*} is agent 2's maximizing bundle on the triangle, we also need the equality $f_2(\theta^*) = \pi/3$. Altogether, we have defined (uniquely) the function f_2 at θ^* and in the interval $[\bar{\theta}, \pi/3]$. We now complete its definition in any way we want, provided of course that we respect its continuity and monotonicity. The relation defined by f_2 solves our problem.

8. Manipulation of the Lindahl correspondence

We now turn to the analysis of various issues concerning the existence of allocation rules satisfying certain criteria of desirability. Although we mainly discuss strategic issues, we also address several normative questions.

Figure 23a shows how an agent, agent 1, can manipulate the Lindahl correspondence to his advantage if the other agent behaves honestly, and how to identify his optimal misrepresentation. The allocation that results is the point agent 1 prefers on agent 2's offer curve *truncated* by the line parallel to the left side of the triangle passing through the endowment point. [In the private good case, no truncation occurs and we would

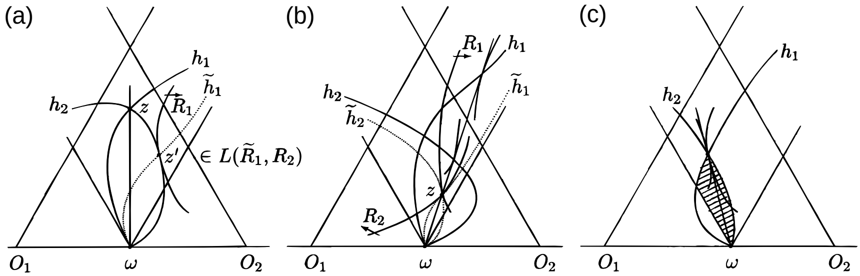

Figure 23: **Manipulability of the Lindahl correspondence.** (a) Identifying the best allocation that agent 1 can reach by optimally manipulating the Lindahl correspondence. (b) Showing how a point z in the truncated lens defined by the true offer curves is an equilibrium allocation of the Lindahl manipulation game. The strategies $\tilde{h}_1$ and $\tilde{h}_2$ rationalize this point as equilibrium allocation. (c) Set of equilibrium allocations of the Lindahl manipulation game.

obtain the point agent 1 prefers on agent 2's offer curve, his monopoly point.][15]

Figure 23b shows the construction of the set of equilibrium allocations of the manipulation game associated with the Lindahl correspondence (Thomson 1979; this result is a counterpart of a result obtained by Hurwicz 1978 for the private good case): identify the lens defined by the true indifference curves passing through the initial allocation and truncate it by lines parallel to the slanted sides of the triangle emanating from the initial allocation. [Again, in the private good case, there is no counterpart of this truncation.] Given any point in the truncated lens such as z, the figure shows how two offer curves, $\tilde{h}_1$ and $\tilde{h}_2$, can be drawn whose point of intersection is for each agent the point he truly prefers on the other agent's announced offer curve. Since only points of intersection of the announced offer curves can be Lindahl allocations, we have a Nash equilibrium. We leave it to the reader to show that at no point outside of the truncated lens can this double maximization be satisfied.

For restricted classes of economies, the set of equilibrium allocations may be quite different from what it is when only the classical properties are imposed on preferences. For instance, suppose that it is known that agents have Cobb–Douglas preferences but the exact value of the Cobb–Douglas parameter of each agent's map is unknown. Then there is a unique equilibrium allocation, identified in Figure 24 (Thomson 1979).

[15]If the number of agents is large, the benefit from unilateral manipulation increases. [In an economy with only private goods, it is also true that an agent typically benefits from taking into account the impact he has on the equilibrium prices, but as the number of agents increases, this impact decreases.]

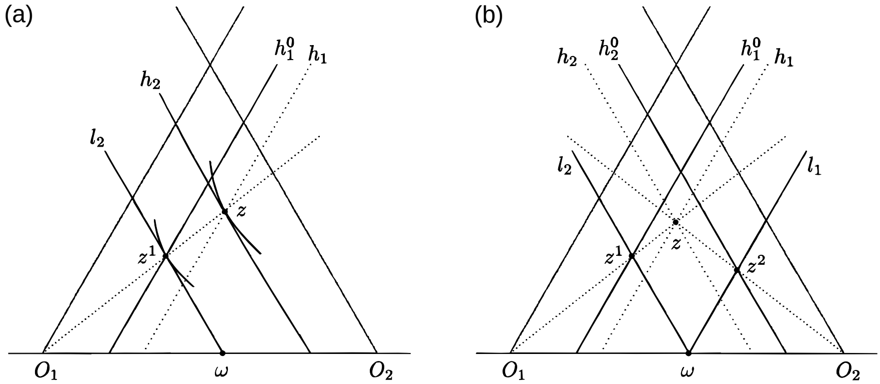

Figure 24: **Manipulation of the Lindahl correspondence in the Cobb–Douglas case.** (a) Identifying the best allocation that agent 1 can reach by optimally manipulating the Lindahl correspondence, given agent 2's Cobb–Douglas announcement. (b) The intersection of the best-reply correspondences.

First, given agent 2's announcement, we determine agent 1's best reply. It is obtained by calculating where agent 1's true expansion path relative to a relative price of the public good equal to 1 ($\pi_1 = 1$) intersects agent 2's announced offer curve, and then choosing a Cobb–Douglas coefficient so that this best point is the resulting Lindahl allocation. The best-response property is obtained at the intersection of the two resulting offer curves.

9. The Nonexistence of Nonmanipulable Correspondences

In light of the manipulability of the Lindahl correspondence, it is natural to ask whether correspondences exist that are immune to this problem. In this section, we show the answer to be mainly negative. Specifically, we look for correspondences such that for each agent, announcing his true preferences is at least as desirable as announcing any other preferences, independently of what his true preferences are and independently of what the other agents announce. To simplify the analysis, we limit our attention to single-valued correspondences.

Formally, the single-valued correspondence φ defined on a domain $\mathcal{E}^N = \mathcal{R}^N \times \mathbb{R}_+^N$ is **strategy-proof** if for all $R \in \mathcal{R}^N$, all $i \in N$, and all $R_i' \in \mathcal{R}$, $\varphi_i(R,\omega_x)\ R_i\ \varphi_i(R_i', R_{-i}, \omega_x)$.[16]

Theorem 1 is an adaptation to the homothetic case of a result established by Ledyard and Roberts (1974) and Groves and Ledyard (1987) for

[16]The notation R_{-i} designates the profile R from which the ith component has been deleted and (R_i', R_{-i}) the profile R in which R_i has been replaced by R_i'.

the quasi-linear case, itself an adaptation for public good economies of a result due to Hurwicz (1979) for the private good case.

THEOREM 1: *On the domain of economies with classical and homothetic preferences, there is no strategy-proof selection from the endowments lower bound and Pareto correspondence.*

Let $\mathcal{R}_{cl,h} \subseteq \mathcal{R}_{cl}$ denote the class of continuous, monotone, convex, and homothetic preferences and $\mathcal{E}^N_{cl,h} = \mathcal{R}^N_{cl,h} \times \mathbb{R}^N_+$ the corresponding domain of economies.

Proof: Let $\varphi : \mathcal{E}^N_{cl,h} \to Z$ be such that $\varphi \subseteq \underline{B}_{end}P$. For the economy $(R, \omega_x) \in \mathcal{E}^N_{cl,h}$ depicted in Figure 25, where $R_1 = R_2$, we have $\underline{B}_{end}P(R, \omega_x) = [a, c] \cup [c, b]$. Suppose that $z = \varphi(R, \omega_x) \in [a, c]$. Now, let agent 1 announce the linear preferences $R'_1 \in \mathcal{R}_{cl,h}$ indicated on the figure. Then, $\underline{B}_{end}P(R'_1, R_2, \omega_x) = [a', b']$. Agent 1, if his true preferences are R_1, finds any point on $[a', b']$ preferable to z. Supposing next that $\varphi(R, \omega_x \in [c, b]$, we would consider a symmetric misrepresentation by agent 2. ■

Theorem 1 raises the question of the existence of *strategy-proof* correspondences on domains of economies with more than two agents. The following negative result does not involve efficiency, but instead a distributional requirement that is stronger than the one of Theorem 1.

THEOREM 2 (Saijo 1991): *On the domain of n-person economies with classical preferences, there is no strategy-proof selection from the augmented-endowments lower bound correspondence.*

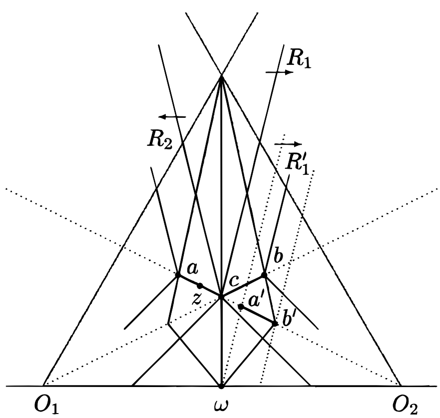

Figure 25: **There is no strategy-proof selection from the endowments lower bound and Pareto correspondence** (Theorem 1).

It is known that on the domain of two-person private good economies with classical preferences, an efficient correspondence is *strategy-proof* if and only if there is a particular agent such that it systematically picks an allocation this agent prefers (Zhou 1991b); such a correspondence is dictatorial. Schummer (1997) shows that this result remains true even on very small subdomains, such as the domain of economies with strictly monotone and linear preferences. A similarly strong result holds on domains of economies with public goods, and here, too, even if in fact preferences are required to belong to the very narrow class of strictly monotone and linear preferences. Indeed in the two-person case, together with efficiency, only the two dictatorial correspondences are admissible (Schummer 1996). For further results on the issue of *strategy-proofs*; see Zhou (1991a).

10. Implementation and the Lindahl and Equal-Cost-Share Correspondences

The theory of implementation was developed to circumvent the problem of manipulation. The theory is concerned with the identification of circumstances in which agents can be led to the allocations that a chosen correspondence would select by confronting them with an appropriately designed "game form." A **game form** is a pair (S, h) of a cross-product $S = S_1 \times \cdots \times S_n$ of **strategy spaces** and an **outcome function** $h\colon S \to Z$. A correspondence φ is (Nash)-**implementable** if there is a game form (S, h) such that for each economy R in the domain of definition of the correspondence, the set of equilibrium allocations of the associated game[17] (S, h, R) coincides with the set of allocations that the correspondence would have selected on the basis of truthful information, $\varphi(R, \omega_x)$. Maskin (1977) shows that for a correspondence to be implementable it should be such that if it chooses a certain allocation for some profile, it should still choose it for any other profile in which the allocation is at least as high as before in the estimation of all agents relative to any other *feasible* allocation. To formally define this property, given $z_i \in \mathbb{R}^2_+$, let $L^t(R_i, z_i)$ denote agent i's lower contour set at z_i truncated by the feasibility constraint: $L^t(R_i, z_i) = \{z' \in Z\colon z_i\ R_i\ z'_i\}$.

Maskin-monotonicity: For all $(R, \omega_x) \in \mathcal{E}^N$, all $R' \in \mathcal{R}^N$, and all $z \in \varphi(R, \omega_x)$, if for all $i \in N$, $L^t(R_i, z_i) \subseteq L^t(R'_i, z_i)$, then $z \in \varphi(R', \omega_x)$.

If R_i and R'_i satisfy the hypotheses of the condition, we say that $\boldsymbol{R'_i}$ **is obtained from** $\boldsymbol{R_i}$ **by a monotonic transformation at** $\boldsymbol{z}$, and we apply the

[17] A game differs from a game form in that it includes information on how agents evaluate outcomes.

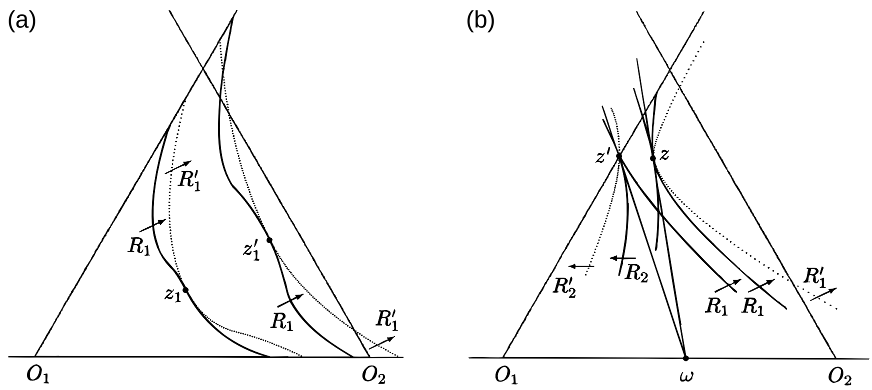

Figure 26: **Maskin-monotonicity.** (a) The preference relation R_1' is obtained from R_1 by a monotonic transformation at z because the lower contour set of R_1' at z_1 contains the lower contour set of R_1 at that point. At z_1', the lower contour sets are not related by inclusion, but when truncated by the feasibility constraint, they are, and this is all we need to conclude that R_1' is obtained from R_1 by a monotonic transformation at z_1'. (b) The Lindahl correspondence is Maskin-monotonic at z, which is interior, but it may violate the property on the boundary of the triangle: z' is a Lindahl allocation for R but not for (R_1, R_2'), a profile obtained from R by subjecting R_2 to a monotonic transformation at z_2'. Indeed, at the only candidate equilibrium prices, agent 2's maximizing bundle is not z_2'.

expression to profiles of preferences if the condition holds for each agent. Figure 26a illustrates the definition and Figure26b shows that at any interior allocation, such as z, and under convexity of preferences the Lindahl correspondence satisfies it. However, the Lindahl correspondence may violate the condition on the boundary, as illustrated in Figure 26b at z'. The Pareto correspondence may also violate the condition at boundary allocations, as shown in Figure 27a at z, and such a violation may occur even if preferences are strictly monotone. [Under this assumption, the Pareto correspondence is *Maskin-monotonic* in private good economies.] If preferences are not convex, violations of *monotonicity* can also occur in the interior of the feasible set (as they do in the private good case; Thomson 1997).

The **constrained Lindahl correspondence**[18] is defined as the Lindahl correspondence except that each agent maximizes his preference relation on the subset of his budget set that is part of some feasible allocation. The allocation z' of Figure 26b is not a Lindahl allocation for the profile (R_1, R_2') as we noted earlier, but it is a constrained Lindahl allocation.

[18]Tian (1988) proposes a different definition of the constrained Lindahl correspondence and shows that it is not monotonic. His definition involves a different truncation.

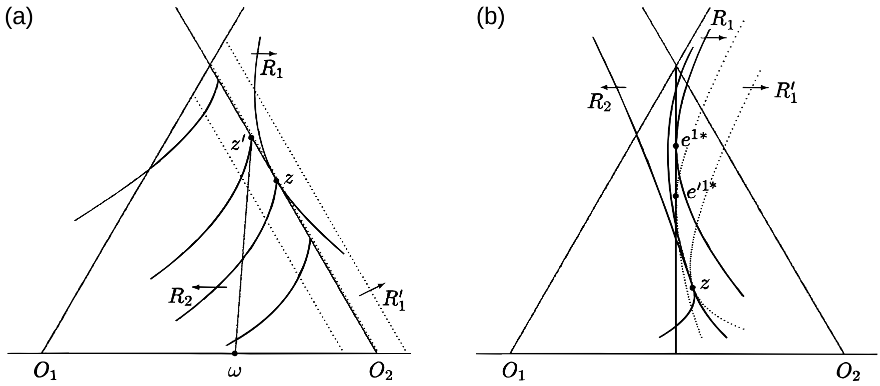

Figure 27: **Maskin-monotonicity of two other correspondences.** (a) The Pareto correspondence may violate Maskin-monotonicity on the boundary of the feasible set even if preferences are strictly monotone. Here, z is efficient for $R = (R_1, R_2)$ but not for (R_1', R_2), a profile obtained from R by subjecting R_1 to a monotonic transformation at z. (b) The preferred-equal-bundle-allocation upper bound and Pareto correspondence is not Maskin-monotonic.

Figure 26a shows that the constrained Lindahl correspondence is not a subcorrespondence of the Pareto correspondence even if preferences are strictly monotone. However, it is a subcorrespondence of the weak Pareto correspondence. It is *Maskin-monotonic*, and in fact it can be described as the smallest *monotonic* correspondence containing the Lindahl correspondence, its **minimal monotonic extension** (Thomson 1997).

Theorem 3, below, brings out the central role played by the Lindahl correspondence when *implementability* is required. It is a counterpart of a variant of a theorem due to Hurwicz (1979) that appears in this form in Thomson (1985). It involves the very mild requirement on a correspondence that if the initial allocation is efficient then the correspondence should select any allocation that is Pareto-indifferent to it.[19] The requirement is satisfied by all of the correspondences defined earlier, and by their extensions to economies with a social endowment, except in that case by the equal-cost-share and Pareto correspondence:

Condition α: *For all* $(R,\omega_x) \in \mathcal{E}^N$, *if* $\omega_x \in P(R,\omega_x)$, *then* $\varphi(R,\omega_x) \supseteq \underline{B}_{end}(R,\omega_x)$.

THEOREM 3 (Thomson 1985): *If a correspondence defined on the classical domain satisfies Condition α and Maskin-monotonicity, then it contains the Lindahl correspondence.*

[19]Two allocations are Pareto-indifferent if all agents are indifferent between their components of it.

Proof: Refer to Figure 28a. Let $\varphi : \mathcal{E}_{cl}^{N} \to Z$ be a correspondence satifying *Condition* α and *Maskin-monotonicity*. Let $(R,\omega_x) \in \mathcal{E}_{cl}^{N}$ be given and $z \in L(R,\omega_x)$, with π as supporting prices. Let $\bar{R}_1 \in \mathcal{R}_{cl}$ be a linear preference relation whose indifference curves are normal to $(1,\pi_1)$, and $\bar{R}_2 \in \mathcal{R}_{cl}$ be a linear preference relation with indifference curves normal to $(1,\pi_2)$. Then, $\omega_x \in P(\bar{R},\omega_x)$. By *Condition* α, $\varphi(\bar{R},\omega_x) \supseteq \underline{B}_{end}(\bar{R},\omega_x)$, so that $z \in \varphi(\bar{R},\omega_x)$. Note that R is obtained from $\bar{R}$ by a monotonic transformation at z. By *Maskin-monotonicity*, $z \in \varphi(R,\omega_x)$. ■

The next theorem, which pertains to economies with a social endowment, states that equal cost shares (equivalently, no-envy) are always forced on agents that may have different preferences by any implementable correspondence satisfying the very natural requirement that if their preferences were the same they should indeed bear equal cost shares. Although one could conceive of requiring that different agents contribute differently to the cost of producing the public good because they derive different benefits from it, implementability actually prevents taking this sort of information into account.

THEOREM 4 (Geanokoplos and Nalebuff, 1988; Moulin 1993; Fleurbaey and Maniquet 1997): *If a correspondence defined on the classical domain of economies with a social endowment satisfies equal treatment of equals and Maskin-monotonicity, it is a subcorrespondence of the equal-cost-shares correspondence.*

Proof: Refer to Figure 28b. Let $\varphi : \mathcal{F}_{cl}^{N} = \mathcal{R}_{cl}^{N} \times \mathbb{R}_{+} \to Z$ be *Maskin-monotonic*. Let $(R,\Omega_x) \in \mathcal{F}_{cl}^{N}$, and suppose that there is $z \in \varphi(R,\Omega_x)$ such that $z_i \neq z_j$ (in the figure, $z_1 \leq z_2$). Let $R_0 \in \mathcal{R}_{cl}$ be a preference

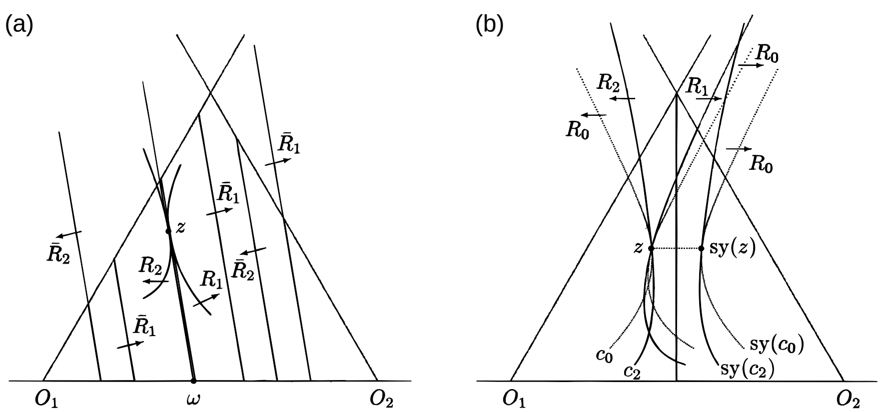

Figure 28: **Two results on implementation.** (a) Theorem 3 shows the central role played by the Lindahl correspondence when implementability is desired. (b) Theorem 4 shows the relevance of equal costs in this context.

relation that is obtained by a Maskin-monotonic transformation of R_1 at z_1 and by a Maskin-monotonic transformation of R_2 at z_2, On the figure, we have measured agent 2's bundle from O_1—from this origin, the bundle is the first component of the allocation $sy(z)$—and drawn the symmetric image of agent 2's indifference curve through z with respect to the vertical line through the top vertex—the curve $sy(c_2)$—to make it clear that a well-behaved map R_0 can indeed be drawn). By *Maskin-monotonicity*, $z \in \varphi(R_0, R_0, \Omega_x)$. This is in violation of *equal treatment of equals*. ■

11. Monotonicity Properties and the Lindahl Correspondence

In this section, we consider requirements having to do with changes in endowments, individual or collective.

Most of the following facts, illustrated in Figures 29, 30, and 31, are noted by Thomson (1987). An additional study of the problem is due to Sertel (1994). Thomson (1987) proves general results concerning the com-

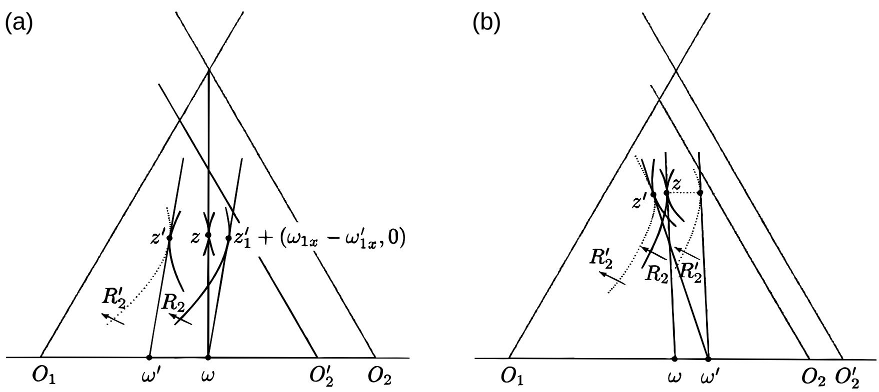

Figure 29: **Two monotonicity properties and the Lindahl correspondence.** The initial allocation is ω and operating the Lindahl correspondence leads to z. (a) Agent 1 withholds the amount $\omega_{1x} - \omega_{1x}'$, which causes the triangle to shrink. Keeping the origin of agent 1's commodity space fixed, agent 2's origin moves to O_2', a point obtained from O_2 by the same translation that takes ω to $\omega' = (\omega_{1x}', \omega_{2x}, 0)$. Agent 2's indifference curve through z_2' is redrawn with O_2 as origin to show that it is compatible with his indifference curve through z_2. (b) Here, agent 1's endowment increases from ω_{1x} to ω_{1x}'. Prices move sufficiently against him for him to lose.

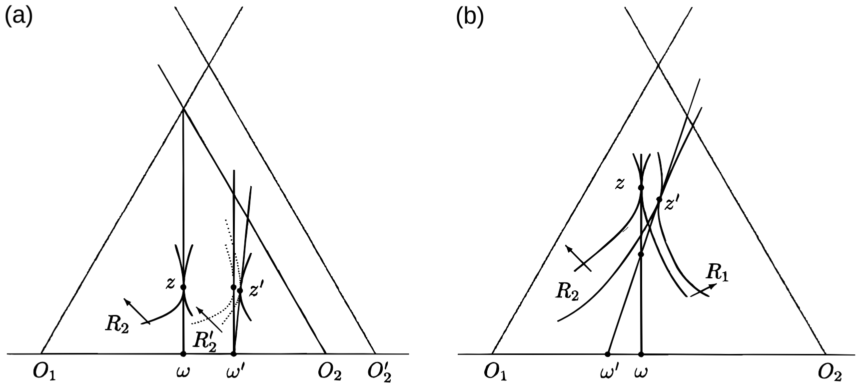

Figure 30: **Two other monotonicity properties and the Lindahl correspondence.** (a) Agent 2 loses due to the increase in agent 1's endowment. Here, $\omega = (\omega_{1x}, \omega_{2x}, 0)$ and $\omega' = (\omega'_{1x}, \omega_{2x}, 0)$ for $\omega'_{1x} > \omega_{1x}$. (b) Agent 1 gains from a transfer of some of his endowment to agent 2. Here, z is a Lindahl allocation from ω and z' is a Lindahl allocation from ω', a new initial allocation obtained by such a transfer.

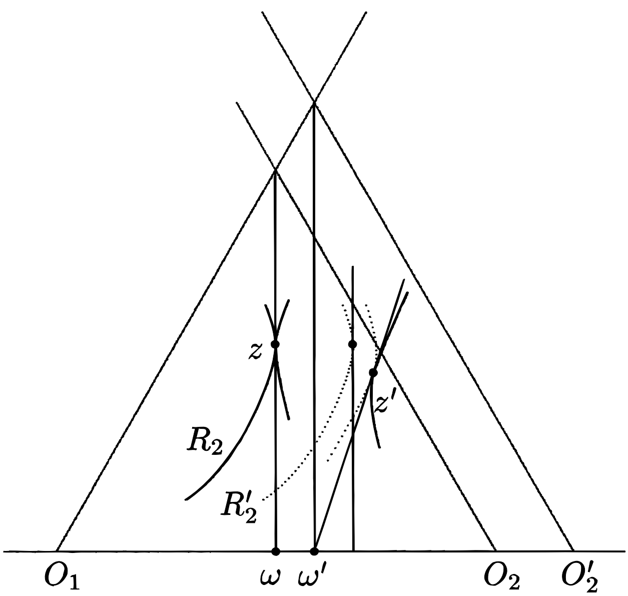

Figure 31: **Monotonicity with respect to increases in the social endowment.** When the social endowment of the private good increases from Ω_x to Ω'_x, and the Lindahl correspondence is operated from equal division, agent 2 loses. Here, $\omega = (\Omega_x/2, \Omega_x/2, 0)$ and $\omega' = (\Omega'_x/2, \Omega'_x/2, 2)$.

patibility of these requirements with efficiency and various distributional criteria.

When the Lindahl correspondence is operated, the following possibilities emerge:

- An agent may gain by withholding part of his endowment (Figure 29a).
- In fact, an agent may gain by destroying part of his endowment. Equivalently, an agent may lose when his endowment increases (Figure 29b).
- An agent may lose when the endowment of the other agent increases (Figure 30a).
- An agent may gain by transferring part of this endowment to the other agent (Figure 30b). Note that, as for the private good case, the possibility of this "transfer paradox" requires multiplicity of the Lindahl allocations from some initial allocation. Indeed the equilibrium price lines have to cross; by choosing the point of intersection as initial allocation, we obtain several equilibrium prices.
- An agent may lose when the social endowment increases (Figure 31).

12. Conclusion

Although the Kolm triangle only pertains to the two-agent case, the case of more than two agents is often not much more complicated, and the intuition the triangle provides suffices to make it understandable. Obviously, a number of issues cannot be illustrated by means of the triangle. It does not accommodate nonlinear technologies, and it does not allow studying how correspondences respond to changes in technologies. Another issue that cannot be studied in the triangle are changes in the population because such changes are of interest only if there may be at least three agents.

References

CAMPBELL, D., and M. TRUCHON (1988) Boundary optima and the theory of public goods supply, *Journal of Public Economics* **35**, 241–249.

CHAMPSAUR, P. (1976) Symmetry and continuity properties of Lindahl equilibria, *Journal of Mathematical Economics* **3**, 19–36.

CONLEY, J., and D. DIAMANTARAS (1996) Generalized Samuelson conditions and wefare theorems for nonsmooth economies, *Journal of Public Economics* **59**, 137–152.

DIAMANTARAS, D. (1992) On equity with public goods, *Social Choice and Welfare* **9**, 141–157.

DIAMANTARAS, D., and S. WILKIE (1996) On the set of Pareto efficient allocations in economies with public goods, *Economic Theory* **7**, 371– 379.

FLEURBAEY, M., and F. MANIQUET (1997) Implementability and horizontal equity require no-envy, *Econometrica* **65**, 1215–1219.

FOLEY, D. (1967) Resource allocation and the public sector, *Yale Economic Essays* **7**, 45–98.

GEANOKOPLOS, T., and B. NALEBUFF (1988) On a fundamental conflict between equity and efficiency. Yale University Mimeo.

GROVES, T., and J. LEDYARD (1987) Incentive-compatibility since 1972; in *Information, Incentives and Economic Mechanisms*, T. GROVES, R. RADNER, and S. REITER, eds. Minneapolis: University of Minnesota Press, 48–111.

HURWICZ, L. (1978) On the interaction between information and incentives in organizations; in *Communication and Control in Society*, K. KRIPPENDORFF, ed. New York: Scientific Publishers Inc., 123–147.

HURWICZ, L. (1979) On allocations attainable through Nash equilibria; in *Aggregation and Revelation of Preferences,* J. J. LAFFONT, ed. Amsterdam: North-Holland, 397–419.

KOLM, S.-C. (1970) *La Valeur Publique*. Paris: Dunod.

LEDYARD, J., and J. ROBERTS (1974) On the incentive problems with public goods, Northwestern University discussion paper.

LINDAHL, E. (1919) *Die Gerechtichkeit der Besteurung*, Lund: Gleerup. (Reprinted as: (1958) Just taxation—A positive solution; in *Classics in the Theory of Public Finance*, R. A. MUSGRAVE and A. T. PEACOK, eds. London: MacMillan.)

MAS-COLELL, A., M. WHINSTON, and J. GREEN (1995) *Microeconomic Theory*. Oxford: Oxford University Press.

MASKIN, E. (1977) Nash equilibrium and welfare optimality, MIT mimeo.

MILLERON, J.-C. (1972) Theory of value with public goods: A survey article, *Journal of Economic Theory* **5**, 419–477.

MOULIN, H. (1990) Fair division under joint ownership: Recent results and open problems, *Social Choice and Welfare* **7**, 149–170.

MOULIN, H. (1993) On the fair and coalitionproof allocation of private goods; in *Frontiers of Game Theory*, K. BINMORE, A. KIRMAN, and P. TANI, eds. Cambridge, MA: MIT Press, 151–163.

SAIJO, T. (1990) Boundary optima and the theory of public goods supply, *Journal of Public Economics* **42**, 213–217.

SAIJO, T. (1991) Incentive-compatibility and individual rationality in public good economies, *Journal of Economic Theory* **55**, 203–212.

SAMUELSON, P. (1955) The pure theory of public expenditure, *Review of Economic Studies* **46**, 387–389.

SCHUMMER, J. (1996) Strategy-proofness versus efficiency for small domains of preferences over public goods, forthcoming in *Economic Theory*.

SCHUMMER, J. (1997) Strategy-proofness versus efficiency on restricted domains of exchange economies, *Social Choice and Welfare* **14**, 47–56.

SERTEL, M. (1994) Manipulating Lindahl equilibrium via endowments, *Economics Letters* **46**, 167–171.

THOMSON, W. (1979) The equilibrium allocation of Walras and Lindahl manipulation games, University of Minnesota discussion paper.

THOMSON, W. (1985) Lecture notes on manipulation and implementation.

THOMSON, W. (1987) Monotonic allocation rules in public good economies, revised April 1998.

THOMSON, W. (1997) Monotonic extensions on economic domains, mimeo, forthcoming in *Review of Economic Design*.

THOMSON, W. (1998) The Theory of Fair Allocation, book manuscript.

TIAN, G. (1988) On the constrained Walrasian and Lindahl correspondences, *Economics Letters* **26**, 299–303.

VARIAN, H. (1974) Equity, envy, and efficiency, *Journal of Economic Theory* **9**, 63–91.

ZHOU, L. (1991a) Impossibility of strategy-proof mechanisms in economies with public goods, *Review of Economic Studies* **58**, 107–119.

ZHOU, L. (1991b) Inefficiency of strategy-proof allocation mechanisms in pure exchange economies, *Social Choice and Welfare* **8**, 247–254.