Existence, Computation, and Applications of Equilibrium

2.1 Art and Bart each sell ice cream cones from carts on the boardwalk in Atlantic City. Each day they independently decide where to position their carts on the boardwalk, which runs from west to east and is exactly one mile long. Let's use x_A and x_B to denote how far (in miles) each cart was positioned yesterday from the west end of the boardwalk; and we'll use x'_A and x'_B to denote how far from the west end the carts are positioned today. Art always positions his cart as far from the boardwalk's west end as Bart's cart was *from the east end* yesterday – i.e., $x'_A = 1 - x_B$. Bart always looks at x_A (how far Art was from the west end yesterday) and then positions his own cart $x'_B = x_A^2$ miles from the west end today. Apply Brouwer's Fixed Point Theorem to prove that there is a stationary pair of locations, (x_A^*, x_B^*) – i.e., a location for each cart today that will yield the same locations again tomorrow. (This problem can be easily solved by other means – in fact, it's easy to calculate the stationary configuration. But the exercise here is to use Brouwer's Theorem.)

2.2 Two Manhattan pretzel vendors must decide where to locate their pretzel carts along a given block of Fifth Avenue. Represent the "block of Fifth Avenue" by the unit interval $I = [0,1] \subseteq \mathbb{R}$ – *i.e.*, each vendor chooses a location $x_i \in [0,1]$. The profit π_i of vendor i depends continuously on *both* vendors' locations – *i.e.*, the function $\pi_i : I \times I \to \mathbb{R}$ is continuous for $i = 1, 2$. Furthermore, each π_i is strictly concave in x_i.

Define an equilibrium in this situation to be a joint action $\widehat{x} = (\widehat{x}_1, \widehat{x}_2) \in I^2$ that satisfies both

$$\forall x_1 \in I : \pi_1(\widehat{x}) \geqq \pi_1(x_1, \widehat{x}_2) \qquad \text{and} \qquad \forall x_2 \in I : \pi_2(\widehat{x}) \geqq \pi_2(\widehat{x}_1, x_2).$$

In other words, an equilibrium consists of a location for each vendor, with the property that each one's location is best for him given the other's location.

(a) Prove that an equilibrium exists.

(b) Generalize this result to situations in which each π_i is merely quasiconcave – *i.e.*, the set $U_i(x) := \{x' \in I^2 \mid \pi_i(x') \geqq \pi_i(x)\}$ is convex for each i and for every $x \in I^2$.

2.3 In doing applied microeconomics you often have to compute equilibria of models that don't have closed-form solutions. The computation therefore must be done by iterative numerical methods. That's what you'll do in this exercise, for the two-person two-good Cobb-Douglas example we analyzed in the first lecture, where each consumer's utility function has the form $u(x,y) = x^{\alpha}y^{1-\alpha}$, and where $\alpha_1 = 7/8$, $\alpha_2 = 1/2$, and $(\mathring{x}_1, \mathring{y}_1) = (40, 80)$, $(\mathring{x}_2, \mathring{y}_2) = (80, 40)$. The iterative computation is pretty straightforward, because there are only two goods and the demand functions have simple closed-form solutions. Moreover, the equilibrium itself has a closed-form solution, so you can also have your program compute the equilibrium prices directly and then check whether your iterative program converges to the correct equilibrium prices.

Specifically, you are to use a spreadsheet program such as Excel, or a programming language such as C+ or Pascal, to compute the path taken by prices and excess demands in the example, assuming that prices adjust according to the transition function in the Gale-Nikaido-Arrow & Hahn proof of existence of equilibrium:

$$f(p) = \frac{1}{\sum_{k=1}^{l}[p_k + M_k(p)]}[p + M(p)]$$

where $M_k(p) = \max(0, \lambda z_k(p))$ for each good k.

The proof did not actually require a λ — *i.e.*, we could assume that $\lambda = 1$ — but with $\lambda = 1$ the iterative process defined by this transition function does not converge for the Cobb-Douglas example, as you can verify once you've created your computational program. You'll find that to achieve convergence you'll need to use a λ equal to about .02 or smaller. Recall, too, that the *proof* does not actually apply to the Cobb-Douglas example, because demands are not defined for the price-lists (1,0) and (0,1). For the same reason, you can't start the iterative process off using either of these as the initial price-lists, because the "next p" defined by $f(p)$ won't be well-defined.

You will of course have to use specific parameter values for the two consumers' utility functions and endowment bundles. With a small enough value for λ, the process will converge for just about any parameter values and any strictly positive initial prices. Of course, when you run your program you should note whether it does converge to the equilibrium price-list.

Plot by hand the price-line and the chosen bundles in the Edgeworth Box for several iterations of the process, or better yet, use our Edgeworth Box applet. Note that if the prices aren't sufficiently close to the equilibrium prices, the chosen bundles may not lie within the confines of the box. This is an important point to understand: each individual consumer simply takes the prices as given and chooses his or her best bundle within the resulting budget set. The consumer takes no account of the total resources available, nor of the other consumers' preferences or choices, because the consumer isn't assumed to have that information.

2.4 There are two goods and n consumers, indexed $i = 1, \ldots, n$. Each consumer has an increasing linear preference – *i.e.*, each consumer's preference is described by a utility function of the form

$$u^i(x_1, x_2) = a^i x_1 + b^i x_2,$$

where a^i and b^i are positive numbers. No production of either good is possible, but each consumer owns positive amounts of each good.

(a) Prove that this economy has a Walrasian equilibrium.

(b) Is the equilibrium price ratio unique? Is the equilibrium allocation unique?

Helpful Hint: Look for ways to make this problem tractable. For example, it might be helpful to index the consumers according to the slopes of their indifference curves – *e.g.*, the flattest as $i = 1$, the next flattest as $i = 2$, and so on. Also, do you need both preference parameters? And it might be easier to work it out first for $n = 2$ and perhaps with each consumer owning the same amount of each good.

You'll probably find it helpful to use the following two theorems on the sum and composition of correspondences that have closed graphs:

Theorem: If Y is compact and the correspondences $f : X \twoheadrightarrow Y$ and $g : X \twoheadrightarrow Y$ both have closed graphs, then the sum $f + g$ also has a closed graph, where $f + g$ is the correspondence defined by

$$(f + g)(x) := \{ \, y_1 + y_2 \in Y \mid y_1 \in f(x) \quad \text{and} \quad y_2 \in g(x) \, \}.$$

Theorem: If Y and Z are compact and the correspondences $f : X \twoheadrightarrow Y$ and $g : Y \twoheadrightarrow Z$ both have closed graphs, then the composition $f \circ g$ also has a closed graph, where $f \circ g$ is the correspondence defined by

$$(f \circ g)(x) := g(f(x)) := \{ \, z \in Z \mid \exists y \in Y : y \in f(x) \,\&\, z \in g(y) \, \} = \bigcup \{ \, g(y) \mid y \in f(x) \, \}.$$

2.5 In our proof of the existence of a Walrasian equilibrium, the following sentences appear: "We know that $\widehat{\zeta}$ has a closed graph and is non-empty-valued and convex-valued, and it is easy to show that μ has the same properties. Therefore so does f, and Kakutani's Theorem therefore implies that f has a fixed point." For this exercise, use the definitions of $\widehat{\zeta}$, μ, and f given in the existence proof. The following proofs are all elementary: the key is understanding the concepts of correspondence, a closed set, a convex set, and the product of two sets. The point of this exercise is to work with those concepts.

(a) If $l = 2$, then μ can be written as follows:

$$\mu(z_1, z_2) = \begin{cases} S, & if z_1 = z_2 \\ \{(1,0)\}, & if z_1 > z_2 \\ \{(0,1)\}, & if z_1 < z_2 \end{cases}$$

For this $l = 2$ case, prove that μ has a closed graph.

(b) Write a detailed definition of μ for the case $l = 3$, like the one above for $l = 2$.

(c) It's obvious in (a) and (b) – assuming you've written (b) correctly – that μ is convex-valued. Give a single proof, for all values of l, that μ is convex-valued.

(d) Prove that if $\widehat{\zeta}$ and μ both have closed graphs, then so does f.

(e) Prove that if $\widehat{\zeta}$ and μ are both convex-valued, then so is f.

2.6 As in Harberger's example, assume that there are two goods produced: product X is produced by firms in the "corporate" sector and product Y by firms in the "non-corporate" sector. Both products are produced using the two inputs **labor** and **capital** (quantities denoted by L and K). Production functions are

$$X = \sqrt{L_X K_X} \qquad \text{and} \qquad Y = \sqrt{L_Y K_Y}.$$

All consumers have preferences described by the utility function $u(X, Y) = XY$. The consumers care only about consuming X and Y, and they supply labor and capital inelastically in the total amounts $L = 600$ and $K = 600$. Let p_X, p_Y, p_L, and p_K denote the prices of the four goods in the economy; assume that $p_L = 1$ always.

(a) What is the Walrasian equilibrium?

(b) Suppose that a 50% tax is imposed on payments to capital in the corporate sector only, and that the government uses the tax proceeds to purchase equal amounts of the output of the two sectors. What will be the new Walrasian equilibrium? How is welfare affected by the tax – are people better off with or without the tax?

2.7 Many applications of microeconomic theory use the concept of a representative consumer. In order for this concept to be meaningful, as we've seen, the economy must satisfy rather special conditions. For this exercise assume there are two consumers, $i = 1, 2$, whose utility functions are $u_i(x_i, y_i) = x_i^{\alpha_i} y_i^{\beta_i}$ and whose initial holdings are $(\mathring{x}_1, \mathring{y}_1)$ and $(\mathring{x}_2, \mathring{y}_2)$. Assume that

$$\frac{\mathring{x}_1}{\mathring{x}_1 + \mathring{x}_2} = \frac{\mathring{y}_1}{\mathring{y}_1 + \mathring{y}_2} = \lambda_1 \qquad \text{and} \qquad \frac{\mathring{x}_2}{\mathring{x}_1 + \mathring{x}_2} = \frac{\mathring{y}_2}{\mathring{y}_1 + \mathring{y}_2} = \lambda_2.$$

According to Eisenberg's Theorem, the market demand function is also the demand function of the ("representative") consumer with initial holdings $(\mathring{x}_1 + \mathring{x}_2,\ \mathring{y}_1 + \mathring{y}_2)$ and with utility function

$$u(x, y) = \max\{\ u_1(x_1, y_1)^{\lambda_1} u_2(x_2, y_2)^{\lambda_2}\ \mid\ x_1 + x_2 = x,\ y_1 + y_2 = y\ \}.$$

Show that $u(x, y) = x^\alpha y^\beta$, where $\alpha = \lambda_1\alpha_1 + \lambda_2\alpha_2$ and $\beta = \lambda_1\beta_1 + \lambda_2\beta_2$.

2.8 Consider the following five sets:

$$\begin{aligned}
A &= \{\ x \in \mathbb{R}^2 \mid x_1, x_2 \geq 0 \quad \text{and} \quad x_1^2 + x_2^2 = 1\ \} \\
B &= \{\ x \in \mathbb{R}^2 \mid x_1, x_2 \geq 0 \quad \text{and} \quad 1 \leq x_1^2 + x_2^2 \leq 2\ \} \\
C &= \{\ x \in \mathbb{R}^2 \mid 1 \leq x_1^2 + x_2^2 \leq 2\ \} \\
D &= \{\ x \in \mathbb{R} \mid 0 \leq x \leq 1\ \} \\
E &= \{\ x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 1\ \}
\end{aligned}$$

(a) Draw a diagram of each set.

(b) To which sets does Brouwer's Fixed Point Theorem apply? (That is, which sets satisfy the assumptions of the theorem?)

(c) Which sets admit a counterexample to Brouwer's Theorem? (That is, for which sets is it possible to define a continuous function f mapping the set into itself for which f has no fixed point?)

(d) For each of the sets you've identified in (c), provide a continuous function f that has no fixed point.

2.9 Two stores, Una Familia and Dos Hijos, are in the same neighborhood and compete in selling a particular product. Every Tuesday, Thursday, and Saturday Dos Hijos changes its posted price p_2 in response to the price p_1 Una Familia charged on the previous day, according to the continuous function $p_2 = f_2(p_1)$. Every Wednesday and Friday Una Familia changes its posted price in response to the price p_2 Dos Hijos charged on the previous day, according to the continuous function $p_1 = f_1(p_2)$. The stores are closed on Sunday; on Monday Una Familia responds to Dos Hijos's preceding Saturday price, also according to $f_1(\cdot)$. Una Familia cannot sell any units at a price above $\bar{p}_1$, no matter what price Dos Hijos charges, so Una Familia never charges a price higher than $\bar{p}_1$. Similarly, Dos Hijos never charges a price higher than $\bar{p}_2$. Prove that there is an "equilibrium" pair of prices (p_1^*, p_2^*) — prices that can persist day after day, week after week, with neither store changing its price.

2.10 (Bewley) A securities analyst publishes a forecast of the prices of n securities. She knows that the prices p_k of the securities are influenced by her forecast according to the continuous function $(p_1, \ldots, p_n) = f(q_1, \ldots, q_n)$, where q_k is her forecast of the price p_k. Whatever prices she forecasts, none of the realized prices ever exceeds Q — *i.e.*, there is a (large) number Q such that $\forall \mathbf{q} \in \mathbb{R}^n : \; f_k(\mathbf{q}) \leqq Q$ for $k = 1, \ldots, n$.

(a) Prove that there exists a forecast $\mathbf{q}^* = (q_1^*, \ldots, q_n^*)$ that will turn out to be perfectly accurate.

(b) The analyst can write down the functions $f_k(q_1, \ldots, q_n)$ for every k, but she can't solve the system of equations $f(\mathbf{q}) = \mathbf{q}$ analytically. Describe a method by which she might be able to arrive at an accurate forecast.

2.11 This exercise builds on Exercise 2.3. Replace the two Cobb-Douglas consumers of #2.3 with consumers whose utility functions have the form $u(x_1, x_2) = 2\sqrt{\alpha x_1} + 2\sqrt{\beta x_2}$.

(a) Derive the consumers' demand functions. You should obtain

$$x_1 = \frac{\alpha}{\beta p_1 + \alpha p_2}\left(\frac{p_2}{p_1}\right) W \qquad \text{and} \qquad x_2 = \frac{\beta}{\beta p_1 + \alpha p_2}\left(\frac{p_1}{p_2}\right) W,$$

where $W = p_1\mathring{x}_1 + p_2\mathring{x}_2$ is the consumer's wealth. Therefore

$$x_1 - \mathring{x}_1 = \frac{1}{\beta p_1 + \alpha p_2}\left(\frac{1}{p_1}\right)\left(\alpha p_2^2\mathring{x}_2 - \beta p_1^2\mathring{x}_1\right) \quad \text{and} \quad x_2 - \mathring{x}_2 = \frac{1}{\beta p_1 + \alpha p_2}\left(\frac{1}{p_2}\right)\left(\beta p_1^2\mathring{x}_1 - \alpha p_2^2\mathring{x}_2\right).$$

(b) Assume that each consumer's α is the same and each consumer's β is the same. Verify that the market excess demand functions for the two goods are the demands of a fictitious "representative consumer" whose utility function is the one given above and whose endowment bundle $(\mathring{x}_1, \mathring{x}_2)$ is the sum of the two actual consumers' endowments: $(\mathring{x}_1, \mathring{x}_2) = (\mathring{x}_1^1, \mathring{x}_2^1) + (\mathring{x}_1^2, \mathring{x}2_2)$, where $\mathring{x}_k^i$ denotes consumer i's endowment of good k. Determine the equilibrium price-ratio.

(c) Now assume that Consumer 1's utility parameters have the values $\alpha^1 = 2$ and $\beta^1 = 1$ and that Consumer 2's are $\alpha^2 = 1$ and $\beta^2 = 1$. Assume that $\mathring{x}_k^i = 40$ for each i and k. In this case the market demand functions are no longer those of a representative consumer, and the equilibrium condition (*viz.* that market excess demand is zero) is a third-degree polynomial equation which would be difficult to solve analytically. (It could be solved numerically; however, if there were more consumers, the equilibrium equation would be even more complicated to solve. And if there were more goods — and thus more price variables and more equations to characterize equilibrium — it would require a very complex numerical procedure to calculate the equilibrium prices directly from the equilibrium equations.) But the computational program you developed in Exercise 2.3 can easily be adapted to calculate the equilibrium price-ratio. How small do you find you must make the price-adjustment parameter λ in order to get the prices to converge? When you get them to converge you should find that the equilibrium price-ratio is approximately $p_1/p_2 = 1.186141$.