Econ 519 Midterm Exam
Fall 2016
Solutions

(1)

(a) $\forall x \in \mathbb{R}: (f+g)(x) = f(x) + g(x)$
$\forall x \in \mathbb{R}, \forall \lambda \in \mathbb{R}: (\lambda f)(x) = \lambda f(x)$.

(b) Let $f_1, f_2 \in \mathcal{P}$: (Note that $\mathcal{P}$ is nonempty)

$f_1(x) = a_0 + a_1 x + \cdots + a_n x^n$
$f_2(x) = b_0 + b_1 x + \cdots + b_m x^m$ ; WLOG assume $m \leq n$.

Define $c_i = a_i + b_i$ $(i = 1, \ldots, m)$
and if $m < n$, define $c_i = a_i$ $(i = m+1, \ldots, n)$.

Then $(f_1 + f_2)(x) = c_0 + c_1 x + \cdots + c_n x^n$, $\forall x \in \mathbb{R}$;
i.e., $f_1 + f_2 \in \mathcal{P}$.

Let $f \in \mathcal{P}$ and $\lambda \in \mathbb{R}$:

$f(x) = a_0 + a_1 x + \cdots + a_n x^n$.

Define $c_i = \lambda a_i$ $(i = 1, \ldots, n)$.

Then
$$(\lambda f)(x) = \lambda f(x) = \lambda(a_0 + a_1 x + \cdots + a_n x^n) = c_0 + c_1 x + \cdots + c_n x^n,$$
i.e., $\lambda f \in \mathcal{P}$.

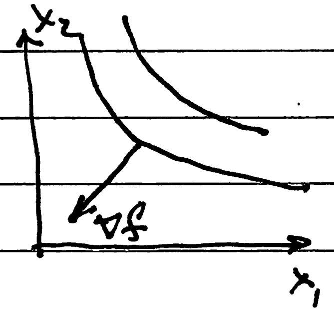
We know that $\mathcal{P} \subseteq \mathcal{F}$, $\mathcal{P} \neq \emptyset$ and if $f_1, f_2 \in \mathcal{P} \Rightarrow f_1 + f_2 \in \mathcal{P}$
and $\lambda \in \mathbb{R}, f \in \mathcal{P} \Rightarrow \lambda f \in \mathcal{P}$,

then $\mathcal{P}$ is a vector subspace of $\mathcal{F}$.

(2) $f : \mathbb{R}^2_{++} \to \mathbb{R}$

$$f(x_1, x_2) = \frac{1}{x_1} + \frac{1}{x_2} = x_1^{-1} + x_2^{-1}$$

$$f_1 = -x_1^{-2} = -\frac{1}{x_1^2}, \quad f_{22} = -x_2^{-2} = -\frac{1}{x_2^2}$$



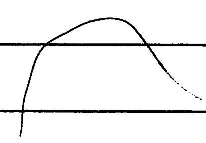

$$\frac{f_1}{f_2} = \frac{1/x_1^2}{1/x_2^2} = \frac{x_2^2}{x_1^2} = \left(\frac{x_2}{x_1}\right)^2$$

$$\nabla f = \left(-\frac{1}{x_1^2}, -\frac{1}{x_2^2}\right)$$

$$f_{11} = 2x_1^{-3}, \quad f_{22} = 2x_2^{-3}, \quad f_{12} = f_{21} = 0$$

$$H = \begin{bmatrix} 2x_1^{-3} & 0 \\ 0 & 2x_2^{-3} \end{bmatrix} = \begin{bmatrix} \frac{2}{x_1^3} & 0 \\ 0 & \frac{2}{x_2^3} \end{bmatrix}$$

$$\left.\begin{array}{l} |H| = 4x_1^{-3}x_2^{-3} = \dfrac{4}{(x_1 x_2)^3} > 0 \\ f_{11} > 0, \; f_{22} > 0 \end{array}\right\} \therefore f \text{ IS CONVEX}$$



(3) LET $S$ BE A CONVEX SUBSET OF $\mathbb{R}^n$; LET $f: S \to \mathbb{R}$; LET $a, b, c \in \mathbb{R}$ WITH $a > 0$; AND $\forall x \in S$ LET $g(x) = af(x) + b$. THEN $f$ IS CONCAVE IF AND ONLY IF $g$ IS CONCAVE.

PROOF:

ASSUME THAT $f$ IS CONCAVE, AND LET $x, x' \in S$ AND $\lambda \in (0,1)$. THEN

$$\begin{aligned} g((1-\lambda)x + \lambda x') &= af((1-\lambda)x + \lambda x') + b \\ &\geqq a[(1-\lambda)f(x) + \lambda f(x')] + b \\ &= (1-\lambda)af(x) + \lambda af(x') + (1-\lambda)b + \lambda b \\ &= (1-\lambda)[af(x) + b] + \lambda[af(x') + b] \\ &= (1-\lambda)g(x) + \lambda g(x'). \end{aligned}$$

THEREFORE $g$ IS CONCAVE.

SINCE $a > 0$, WE HAVE $af(x) = g(x) - b$, i.e., $f(x) = \frac{1}{a}g(x) - \frac{b}{a} = \alpha g(x) + \beta$ FOR $\alpha = \frac{1}{a}$, $\beta = -\frac{b}{a}$. THE ABOVE RESULT THEN SAYS THAT IF $g$ IS CONCAVE, SO IS $f$. ||

4. $f(x_1, x_2) = \frac{1}{2}x_1^2 - \frac{1}{2}x_2^2 + x_1x_2 - x_1 + x_2$.

(a) $f_1 = x_1 + x_2 - 1$, $f_2 = -x_2 + x_1 + 1$ $\quad \nabla f = (x_1+x_2-1, x_1-x_2+1)$

$f_{11} = 1$, $f_{22} = -1$, $f_{12} = f_{21} = 1$ $\quad D^2 f = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

(b) $f_1 = 0: \; x_1 + x_2 - 1 = 0$ $\quad$ THE ONLY SOLUTION IS $x_1 = 0, x_2 = 1$.

$f_2 = 0: \; x_1 - x_2 + 1 = 0$ $\quad \bar{x} = (0, 1)$ AND $f(\bar{x}) = \frac{1}{2}$.

(c) $P_2(\Delta x, \bar{x}) = f(\bar{x}) + \nabla f(\bar{x})\Delta x + \frac{1}{2}\Delta x\, H(\bar{x})\Delta x$

$= \frac{1}{2} + 0 + \frac{1}{2}\Delta x \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\Delta x$, BECAUSE $\nabla f(\bar{x}) = (0,0)$

$= \frac{1}{2} + \frac{1}{2}\left[(\Delta x_1)^2 + 2\Delta x_1 \Delta x_2 - (\Delta x_2)^2\right]$

$= \frac{1}{2} + \frac{1}{2}(\Delta x_1)^2 + \Delta x_1 \Delta x_2 - \frac{1}{2}(\Delta x_2)^2$.

(d) $\bar{x}$ IS A CRITICAL POINT OF $f$, SO WE NEED TO CHECK THE SECOND-ORDER CONDITIONS AT $\bar{x}$:

$|D^2 f(\bar{x})| = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$, SO $D^2 f(\bar{x})$ IS INDEFINITE

AND THEREFORE $\bar{x}$ IS NEITHER A MAX OR MIN.

(e) $G(x_1, x_2) = x_1 + x_2 = 2$; $\nabla G = (1, 1)$.

FOC: $\nabla f = \lambda \nabla G$, i.e.,

$f_1 = \lambda G_1: \; x_1 + x_2 - 1 = \lambda$

$f_2 = \lambda G_2: \; x_1 - x_2 + 1 = \lambda$

$2 - 1 = \lambda$; $\lambda = 1$; $\therefore \bar{x}_1 = \bar{x}_2 = 1$.

$x_1 + x_2 - 1 = x_1 - x_2 + 1$; i.e.,

$x_2 - 1 = 1 - x_2$; $\therefore x_2 = 1, x_1 = \lambda$

SINCE $x_1 + x_2 = 1$, WE HAVE $x_1 = 1$.

$\nabla f(\bar{x}) = (1, 1)$, $\nabla G = (1, 1)$, $\lambda = 1$. $\quad$ SO $\bar{x} = (1, 1)$.

THE SECOND-ORDER CONDITIONS FOR $\bar{x}$ TO BE A LOCAL MAX OR MIN OF $f$ s.t. $G(x) = 2$:

SINCE $m = 1$ AND $n = 2$, WE NEED TO CHECK THE SIGN OF ONLY $|\mathbb{B}|$, WHERE $\mathbb{B}$ IS THE HESSIAN MATRIX OF $f$ AT $\bar{x}$, BORDERED BY $\nabla G(\bar{x})$:

$$\mathbb{B} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}; \quad |\mathbb{B}| = 1 + 1 - 1 + 1 = 2,$$

SO THE QUADRATIC FORM $\Delta x H(x) \Delta x$ IS NEGATIVE DEFINITE AT $\bar{x}$, SUBJECT TO $x_1 + x_2 = 2$, AND $\bar{x}$ IS THEREFORE A LOCAL MAXIMUM OF $f$ SUBJECT TO $x_1 + x_2 = 2$.

NOTE THAT THE BORDERED MATRIX $\mathbb{B}$ IS THE SAME AT EVERY $x$ ON THE CONSTRAINT, SO THE QUADRATIC FORM $\Delta x H(x) \Delta x$, RESTRICTED TO $\Delta x$ THAT KEEP $x$ ON THE CONSTRAINT AND TO $x$ ON THE CONSTRAINT, IS NEGATIVE DEFINITE. THEREFORE $f$, RESTRICTED TO THE CONSTRAINT, IS CONCAVE (ACTUALLY, STRICTLY CONCAVE), SO $\bar{x}$ IS A <u>GLOBAL</u> MAXIMUM SUBJECT TO THE CONSTRAINT.

(f) $G(x_1,x_2) = x_1 - x_2 = 2$: $\qquad \nabla G = (1,-1)$

FOC: $\nabla f = \lambda \nabla G$, i.e.,

$$\left.\begin{array}{l} f_1 = \lambda G_1: \quad x_1 + x_2 - 1 = \lambda \\ f_2 = \lambda G_2: \quad x_1 - x_2 + 1 = -\lambda \end{array}\right\} \begin{array}{l} x_1 = 0, \therefore x_2 = -2, \ \lambda = -3. \\ \bar{x} = (0,-2). \end{array}$$

SECOND-ORDER CONDITIONS FOR $\bar{x}$ TO BE A LOCAL MAX OR MIN s.t. $G(x) = 2$:

WE AGAIN NEED TO CHECK THE SIGN OF ONLY $|\mathbb{B}|$:

$$\mathbb{B} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}; \quad |\mathbb{B}| = -1-1-1+1 = -2,$$

SO $\Delta x H(x) \Delta x$ IS POSITIVE DEFINITE SUBJECT TO THE RESTRICTION THAT $\Delta x$ LEAVES US ON THE CONSTRAINT $G(x) = 2$. THEREFORE $\bar{x}$ IS A LOCAL MINIMUM OF $f$ SUBJECT TO $G(x) = 2$. AND JUST AS IN (e), $|\mathbb{B}|$ IS CONSTANT ON THE CONSTRAINT, SO $\Delta x H(x) \Delta x$ IS POSITIVE DEFINITE SUBJECT TO $G(x) = 2$, AND THEREFORE $\bar{x}$ IS A GLOBAL MINIMUM SUBJECT TO $G(x) = 2$.

NOTE THAT (e) AND (f) CAN ALSO BE SOLVED BY CONVERTING THE FUNCTION $f$ INTO A FUNCTION OF ONE VARIABLE: IN (e) WE HAVE $x_2 = 2 - x_1$, SO WE CAN DEFINE $g(x_1) = f(x_1, 2 - x_1)$, A QUADRATIC FUNCTION OF $x_1$, FOR WHICH IT'S EASY TO OBTAIN $g'(x_1)$ AND $g''(x_1)$. THE EXAM SAID, HOWEVER, THAT YOU WERE TO USE THE CONSTRAINED MAX/MIN CONDITIONS. BUT YOU COULD HAVE USED THIS METHOD TO CHECK YOUR SOLUTION.