

## Introductory Notes on Demand Theory

(The Theory of Consumer Behavior, or Consumer Choice)

This brief introduction to demand theory is a preview of the first part of Econ 501A, but it also serves as a prototype or template for other models of decision-making we'll develop in Econ 501A, such as the theory of the firm, the theory of decision-making under uncertainty, etc. Demand theory is used heavily in Econ 501B as well, right from the beginning.

How is demand theory a prototype for other models of decision-making? Demand theory has the *same structure* as the other models, and we'll ask the *same questions* and use the *same techniques* in demand theory as in subsequent decision models.

### The elements of demand theory:

- (1) Alternatives
- (2) Constraints
- (3) A criterion for choosing (*e.g.*, an objective function, a preference ordering, etc.)

It's this alternatives/constraints/criterion structure that's common to models of decision-making. Demand theory provides a familiar example.

### (1) Alternatives:

The **alternatives** in a consumer's choice problem are **consumption bundles**, *i.e.*, amounts of various goods or **commodities**. Suppose there are  $n$  goods. We represent the alternatives algebraically and geometrically:

**Algebra:** Each bundle is an  $n$ -tuple or **list** of  $n$  non-negative numbers,

$$\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}_+^n.$$

**Geometry:** Each bundle is a point in  $\mathbb{R}_+^n$ , a subset of the linear space  $\mathbb{R}^n$ .

**Reality:** Connecting the model to reality requires one to decide how to divide things up into commodities, which also determines how many commodities the model will have. For example, are hamburger and steak two distinct commodities, or are they both just beef? Is gasoline at Speedway & Park the same commodity as gasoline at Exit #248 on I-10, or in San Diego? Another example: there will be applications in which some components of the bundles will have to be restricted to discrete quantities.

As we develop a model of consumer choice (as with any model) we have to pay attention, as above, to the relation between the formal model (the symbolic or algebraic model and the corresponding geometry) and the reality the model is supposed to help us understand (in this case, consumers' actual choices).

## (2) Constraints:

What constraints are there on the alternatives a consumer can choose? We've already introduced  $n$  constraints without mentioning them: by defining the set of alternatives as  $\mathbb{R}_+^n$  we imposed the  $n$  **non-negativity constraints**  $x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0$ . An additional constraint is imposed on the consumer by his budget — what bundles he can afford to buy. If the commodities' prices are  $p_1, p_2, \dots, p_n$  (in dollars per unit, say), and if the consumer's budget is  $w$  dollars, then he can choose only those bundles that satisfy his **budget constraint**

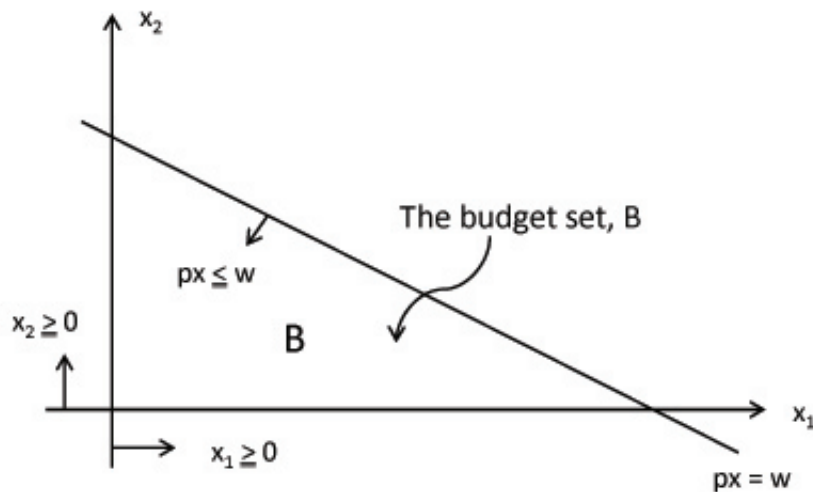
$$\mathbf{p} \cdot \mathbf{x} \leq w; \quad \text{i.e., } p_1x_1 + \dots + p_nx_n \leq w; \quad \text{i.e., } \sum_{k=1}^n p_kx_k \leq w. \quad (BC)$$

The **budget set** or **feasible set** is the set of all bundles  $\mathbf{x}$  that satisfy (BC) and the non-negativity constraints as well:

$$\begin{aligned} B &= \{ \mathbf{x} \in \mathbb{R}_+^n \mid \mathbf{p} \cdot \mathbf{x} \leq w \} \\ &= \{ \mathbf{x} \in \mathbb{R}^n \mid \mathbf{p} \cdot \mathbf{x} \leq w \ \& \ x_i \geq 0, \forall i \}. \end{aligned}$$

Because  $\mathbf{p} \cdot \mathbf{x} \leq w$  is a linear inequality, the budget set is a half-space in  $\mathbb{R}^n$  bounded by the hyperplane  $\mathbf{p} \cdot \mathbf{x} = w$ , truncated by  $x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0$ , or equivalently, it is the intersection of the  $n + 1$  half-spaces defined by the  $n + 1$  linear inequalities

$$\mathbf{p} \cdot \mathbf{x} \leq w, \ x_1 \geq 0, \ x_2 \geq 0, \ \dots, \ x_n \geq 0.$$



### (3) Criterion, or objective:

Here in this preview we will assume that the consumer has a **utility function**  $u : \mathbb{R}_+^n \rightarrow \mathbb{R}$  which tells him (or us) which bundles he likes more and which ones he likes less. We interpret

$u(\tilde{\mathbf{x}}) > u(\mathbf{x})$  to mean he **prefers** bundle  $\tilde{\mathbf{x}}$  to bundle  $\mathbf{x}$ ; *i.e.*, if offered a choice between just  $\tilde{\mathbf{x}}$  and  $\mathbf{x}$ , he would choose  $\tilde{\mathbf{x}}$  ;

and

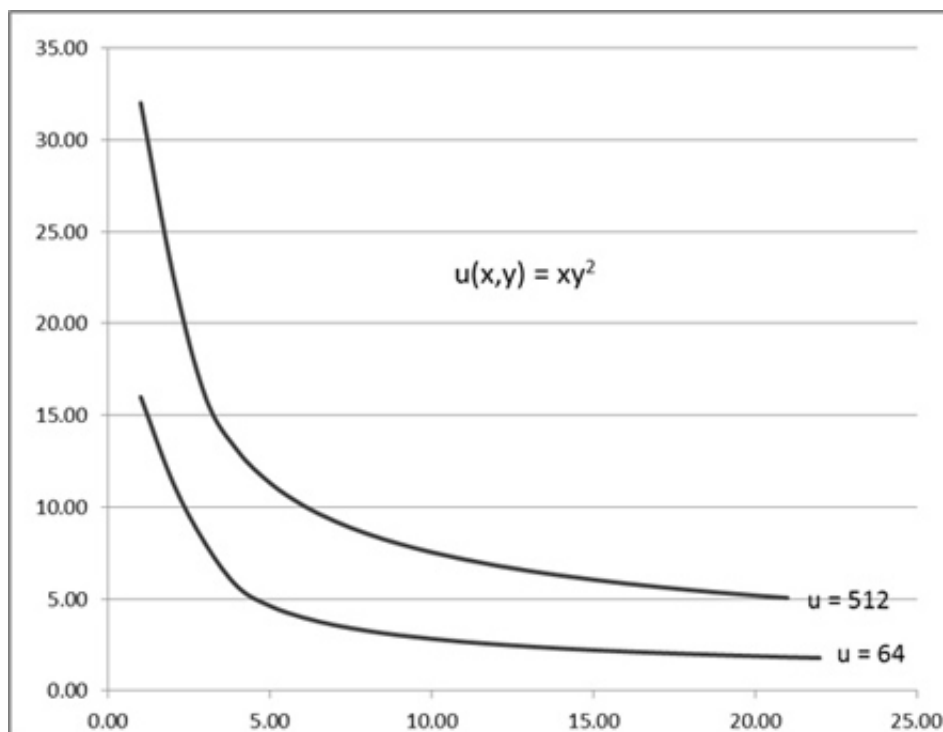
$u(\tilde{\mathbf{x}}) = u(\mathbf{x})$  to mean he is **indifferent** between the two bundles  $\tilde{\mathbf{x}}$  and  $\mathbf{x}$ ; *i.e.*, if offered a choice between just  $\tilde{\mathbf{x}}$  and  $\mathbf{x}$ , he cannot say which he would choose.

Ultimately, we will adopt a weaker and more intuitive assumption — *viz.*, that the consumer's choices can be described by a **preference relation** (essentially an indifference map) that is well-behaved — and we'll determine conditions under which that description of behavior is equivalent to the utility-function description we're using here.

If the function  $u(\cdot)$  is continuous, it can be described geometrically by its **indifference curves**, *i.e.*, its **level curves**. A level curve of  $u(\cdot)$ , as for any function, is a set of the form

$$I_c := \{ \mathbf{x} \mid u(\mathbf{x}) = c \} \quad \text{for some } c \in \mathbb{R}$$

*i.e.*, it's the set of all bundles that yield the utility level  $c$ . Here are two of the level curves of the function  $u(x, y) = xy^2$  :



Note that continuity of  $u$  is not enough to guarantee that its level curves are actually curves: they could be “thick”. The following utility function, for example, is continuous but has a thick indifference curve that includes all the bundles  $(x, y) \in \mathbb{R}_+^2$  that satisfy  $25 \leq xy \leq 36$ :

$$u(x, y) = \begin{cases} xy, & \text{if } xy \leq 25 \\ 25, & \text{if } 25 \leq xy \leq 36. \\ xy - 11, & \text{if } xy \geq 36 \end{cases}$$

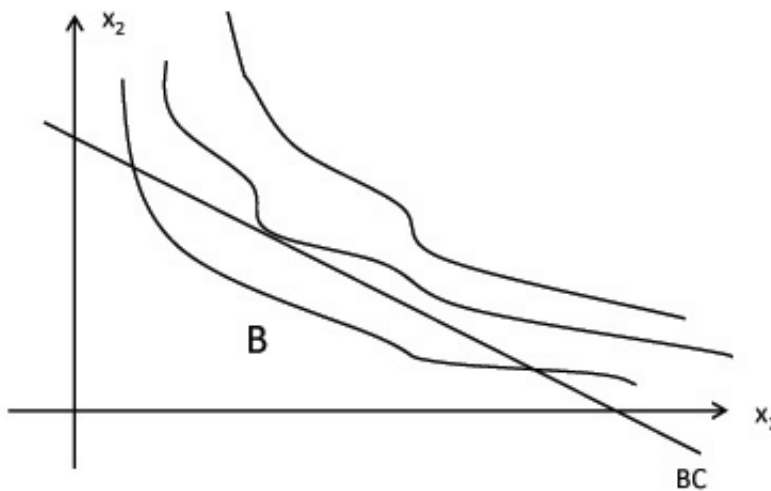
### The Consumer’s Maximization Problem:

We can summarize everything we’ve said about the consumer’s choice problem by simply stating the **consumer’s maximization problem** (which we abbreviate by CMP, or sometimes UMP for **utility maximization problem**):

(CMP) Choose  $\mathbf{x} \in \mathbb{R}_+^n$  to maximize  $u(\mathbf{x})$  subject to  $(BC)$  ;

*i.e.*,

(CMP) Choose  $\mathbf{x} \in \mathbb{R}_+^n$  to maximize  $u(\mathbf{x})$  subject to  $\mathbf{p} \cdot \mathbf{x} \leq w$ .



A solution of (CMP) is a bundle, say  $\hat{\mathbf{x}}$ , that yields the greatest value of  $u(\mathbf{x})$  among all the bundles in the budget set B. Note that there is no guarantee that a solution will exist or that if a solution does exist it will be unique.

## Some Things We Want to Know About the Solution of the CMP:

- (1) Does a solution exist? Or under what conditions does a solution exist?
- (2) Is the solution unique? Or under what conditions is the solution unique?

Note that in the CMP,

- The **consumer** is represented by the **utility function**  $u(\cdot)$ .
- His **environment** is represented by the **parameters**  $p_1, \dots, p_n$  and  $w$ .

- (3) For a specific utility function and parameter values, how can we determine the solution?
- (4) How does the solution change as the consumer's environment changes?

In order to answer (4), what we want to know is the **solution function**, which gives  $\mathbf{x}$  as a function of  $\mathbf{p}$  and  $w$ :

$$\mathbf{x} = f(\mathbf{p}, w),$$

*i.e.*, the consumer's **demand function**, or equivalently, the  $n$  demand functions for the  $n$  goods:

$$\begin{aligned}x_1 &= f_1(\mathbf{p}, w) = f_1(p_1, \dots, p_n, w) \\x_2 &= f_2(\mathbf{p}, w) = f_2(p_1, \dots, p_n, w) \\&\dots \\x_n &= f_n(\mathbf{p}, w) = f_n(p_1, \dots, p_n, w).\end{aligned}$$

The following pages contain several very simple examples in which we obtain the solution for a *specific* utility function, as in (3), and the *solution function* (*i.e.*, the demand function), as in (4).

FOR EXAMPLE, WHEN THERE ARE ONLY TWO GOODS:  
(LET'S USE  $x$  AND  $y$  INSTEAD OF  $x_1$  AND  $x_2$ )

$$(CMP) \max_{(x,y) \in \mathbb{R}_+^2} u(x,y) \text{ s.t. } p_x x + p_y y \leq w$$

FoC: (INTERIOR)

$$u_x(x,y) = \lambda p_x$$

$$u_y(x,y) = \lambda p_y$$

$$p_x x + p_y y = w$$

} WE WANT TO SOLVE THESE THREE EQUATIONS FOR THE THREE VARIABLES  $x, y, \lambda$ :

$$(x, y, \lambda) = f(p_x, p_y, w).$$

NOTE THAT IN THE TWO-GOOD CASE WE CAN ELIMINATE THE  $\lambda$  AND EXPRESS THE ~~TWO~~ TWO MARGINAL CONDITIONS AS A SINGLE EQUATION WITH THE MRS:

$$\frac{u_x}{u_y} = \frac{\lambda p_x}{\lambda p_y} = \frac{p_x}{p_y}; \text{ i.e., } MRS = \frac{p_x}{p_y}$$

↑  
DEPENDS ON  $(x, y)$

LET'S CONSIDER THE UTILITY FUNCTION  
 $u(x, y) = x^2 y$ :

① FOR SPECIFIC VALUES OF THE PARAMETERS, WE CAN SOLVE FOR THE CONSUMER'S CHOSEN BUNDLE.

FOR EXAMPLE, SUPPOSE  $w = 36$  AND  $p_x = 3$ ,  $p_y = 2$ :

$$\max_{(x, y) \in \mathbb{R}_+^2} u(x, y) = x^2 y \quad \text{s.t.} \quad 3x + 2y \leq 36$$

FOC:

$$\begin{aligned} u_x = \lambda p_x: & \quad 2xy = 3\lambda \\ u_y = \lambda p_y: & \quad x^2 = 2\lambda \\ 3x + 2y = & \quad 36 \end{aligned}$$

SOLVING THE EQUATIONS:

$$2 \frac{y}{x} = \frac{3}{2}; \quad 4y = 3x; \quad y = \frac{3}{4}x$$

$$\therefore 3x + 2\left(\frac{3}{4}x\right) = 36$$

$$3x + \frac{3}{2}x = 36$$

$$x = 8, \quad y = 6$$

② BUT WE CAN ALSO SOLVE FOR THE CHOSEN BUNDLE AS A FUNCTION OF THE PARAMETER VALUES — I.E., THE CONSUMER'S DEMAND FUNCTION:

$$\max_{(x, y) \in \mathbb{R}_+^2} u(x, y) = x^2 y \quad \text{s.t.} \quad p_x x + p_y y \leq w$$

FOC:

$$\begin{aligned} u_x = \lambda p_x \\ u_y = \lambda p_y \\ p_x x + p_y y = & \quad w \end{aligned}$$

SOLVING:

$$\frac{u_x}{u_y} = \frac{p_x}{p_y} \quad \text{i.e.,} \quad 2 \frac{y}{x} = \frac{p_x}{p_y}$$

$$\text{i.e.,} \quad 2 p_y y = p_x x$$

$$\therefore p_x x + \frac{1}{2} p_x x = w; \quad \frac{3}{2} p_x x = w$$

$$\text{i.e.,} \quad p_x x = \frac{2}{3} w, \quad p_y y = \frac{1}{3} w$$

$$\therefore x = \frac{2}{3} \left( \frac{w}{p_x} \right), \quad y = \frac{1}{3} \left( \frac{w}{p_y} \right)$$

# EXAMPLES FOR FIRST 501A LECTURE

(CMP)  $\max_{x \in \mathbb{R}_+^n} u(x)$  s.t.  $p \cdot x \leq w$

(UMH) THE CONSUMER CHOOSES A BUNDLE  $x$  THAT IS A SOLUTION OF (CMP).

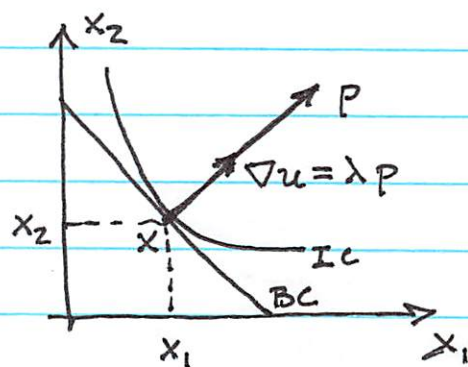
IMPLICATION: (IF  $u(\cdot)$  IS "NICE")

$\exists \lambda > 0 : \nabla u = \lambda p$

i.e.  $\frac{\partial u}{\partial x_k} = \lambda p_k, k=1, \dots, n$

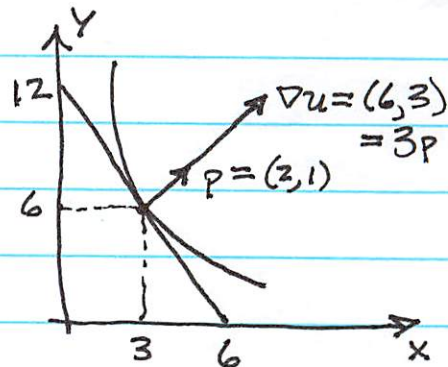
AND ALSO  $p \cdot x = w$ .

FOC  
IC & BC ARE TANGENT AT  $x$



## EXAMPLES:

①  $u(x, y) = xy$   $p_x = 2, p_y = 1, w = 12$   
 $u_x = y, u_y = x; MRS = \frac{u_x}{u_y} = \frac{y}{x}$



FOC:  $\exists \lambda > 0 :$

$u_x = \lambda p_x$  i.e.,  $y = \lambda \cdot 2$   
 $u_y = \lambda p_y$  i.e.,  $x = \lambda \cdot 1$

$p_x x + p_y y = w$  i.e.,  $2x + y = 12 \rightarrow 2x + 2x = 12; 4x = 12$

$\therefore \boxed{x=3, y=6} \therefore \lambda=3.$

NOTE: WHEN  $n=2$  WE CAN USE THE FACT THAT

$MRS = \frac{u_x}{u_y}$  TO WRITE THE FOC THIS WAY:

(M)  $MRS = \frac{p_x}{p_y}$   
 (C)  $p_x x + p_y y = w$

IN EXAMPLE 1:  $\begin{cases} \frac{y}{x} = \frac{2}{1} = 2; \text{ i.e., } y=2x \\ 2x+y=12 \end{cases}$

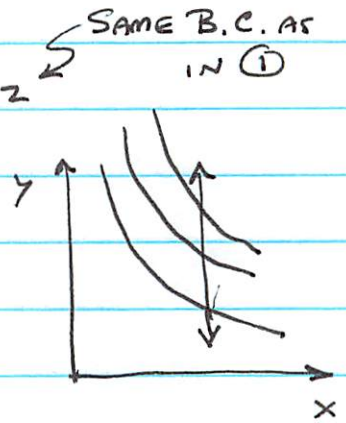
... THEN AS ABOVE.



(2)  $u(x, y) = y + 6 \log x$        $P_x = 2, P_y = 1, W = 12$

$u_x = \frac{6}{x}, u_y = 1; \text{MRS} = \frac{6}{x}$

NOTE THAT MRS DEPENDS ONLY ON  $x$ ,  
SO I-CURVES ARE VERTICAL SHIFTS OF  
ONE ANOTHER.



Foe:

$$\left. \begin{array}{l} u_x = \frac{6}{x} \\ u_y = 1 \end{array} \right\} \begin{array}{l} \frac{6}{x} = \lambda \cdot 2 \\ 1 = \lambda \cdot 1 \end{array} \rightarrow \lambda = 1, x = 3$$

$P_x x + P_y y = W \rightarrow (2)(3) + y = 12; \therefore y = 6$

$x = 3, y = 6$ , SAME AS IN (1);  $\lambda = 1$ .

THIS IS A COINCIDENCE; TRY DIFFERENT VALUES  
FOR  $P_x, P_y$ , AND/OR  $W$  AND YOU WILL FIND  
THAT THE UTILITY FUNCTIONS IN (1) AND (2)  
LEAD TO DIFFERENT BUNDLES BEING CHOSEN.

FOR EXAMPLE, TRY  $P_x = 2, P_y = 1, W = 18$

(PRICES UNCHANGED, BUT BUDGET INCREASED BY 50%)

OR TRY  $P_x = 1.50, P_y = 1, W = 12$

(BUDGET AND  $P_y$  UNCHANGED, BUT  $P_x$  REDUCED BY 25%).

③  $u(x, y) = x^\alpha y^\beta$  (GENERALIZES ① TO GENERAL COBB-DOUGLAS)

WE'LL LEAVE  $P_x, P_y, W$  AS UNSPECIFIED PARAMETERS.

WE WANT TO SOLVE FOR  $x$  AND  $y$  AS FUNCTIONS OF  $P_x, P_y, W$

— i.e., WE WANT TO DERIVE THIS CONSUMER'S

DEMAND FUNCTION ( $x(P_x, P_y, W), y(P_x, P_y, W)$ ).

$$u_x = \alpha x^{\alpha-1} y^\beta = \frac{\alpha}{x} x^\alpha y^\beta = \frac{\alpha}{x} u(x, y)$$

$$u_y = \beta x^\alpha y^{\beta-1} = \frac{\beta}{y} x^\alpha y^\beta = \frac{\beta}{y} u(x, y)$$

$$MRS = \frac{u_x}{u_y} = \frac{\frac{\alpha}{x} u(x, y)}{\frac{\beta}{y} u(x, y)} = \left(\frac{\alpha}{\beta}\right) \frac{y}{x}$$

FOC:

$$MRS = \frac{P_x}{P_y} : \left(\frac{\alpha}{\beta}\right) \frac{y}{x} = \frac{P_x}{P_y} ; \text{ i.e., } \beta P_x x = \alpha P_y y$$

$$P_x x + P_y y = W : P_x x + \frac{\beta}{\alpha} P_x x = W ; \text{ i.e., } \left(1 + \frac{\beta}{\alpha}\right) P_x x = W$$

$$\text{i.e., } \frac{\alpha + \beta}{\alpha} P_x x = W ; \text{ i.e., } P_x x = \frac{\alpha}{\alpha + \beta} W$$

$$\text{SIMILARLY, } P_y y = \frac{\beta}{\alpha + \beta} W.$$

$$\therefore x = \frac{\alpha}{\alpha + \beta} \left(\frac{W}{P_x}\right) \text{ AND } y = \frac{\beta}{\alpha + \beta} \left(\frac{W}{P_y}\right) \leftarrow \text{CONSUMER'S DEMAND FUNCTIONS}$$

CONSTANT EXPENDITURE SHARES