Euclidean Space

This is a brief review of some basic concepts that I hope will already be familiar to you.

There are three sets of numbers that will be especially important to us:
- The set of all real numbers, denoted by $\mathbb{R}$.
- The set of all natural numbers, denoted by $\mathbb{N}$ — *i.e.*, $\mathbb{N} = \{1, 2, 3, \ldots\}$.
- The set of all integers, denoted by $\mathbb{Z}$ — thus, $\mathbb{Z} = \{x \mid x = 0 \text{ or } x \in \mathbb{N} \text{ or } -x \in \mathbb{N}\}$.

For any set X, we can form ***n*-tuples**, or **lists**, of members of X. We write an n-tuple as $(x_1, x_2, \ldots, x_n)$, where each **component** x_i is a member of X. The order of the components matters — *i.e.*, the 3-tuples $(2, 5, 2)$ and $(5, 2, 2)$ are distinct, different 3-tuples of numbers. We therefore sometimes call them **ordered** lists or ordered n-tuples.

Notice that in both 3-tuples above, the number 2 appeared twice. That's allowed: the components of an n-tuple don't have to be distinct. Suppose we're using n-tuples to represent lists of the prices, in dollars, of a gallon of gasoline, a liter of wine, and a gallon of milk, in that order. We don't want to rule out the possibility that both gasoline and milk have the same price, as in the 3-tuple $(2, 5, 2)$. And that would be different than the price-list $(5, 2, 2)$: the order *does* matter.

It's important to remember the distinction between an n-tuple and a set. The facts in the preceding paragraph are just the opposite for sets: the order of a set's elements *doesn't* matter, and elements *can't* repeat. For example, $\{2, 5, 2\}$ makes no sense, and the sets $\{5, 2\}$ and $\{2, 5\}$ are the same set (the order of elements doesn't matter).

For any given set X, we use the notation X^n to denote the set of *all* n-tuples of elements of X. So $\mathbb{R}^n$ is the set of all n-tuples $(x_1, \ldots, x_n)$, where each component x_i is a real number. Similarly, $\mathbb{Z}^n$ is the set of all n-tuples of integers, and $\mathbb{N}^n$ is the set of all n-tuples of natural numbers. The two 3-tuples $(2, 5, 2)$ and $(5, 2, 2)$ are elements of $\mathbb{N}^3$, $\mathbb{Z}^3$, and $\mathbb{R}^3$.

The geometry we associate with the sets $\mathbb{R}^n$ will be an important part of everything we do, and should already be familiar. The set $\mathbb{R}$ is identified geometrically with the "real line," a line with a designated point that represents the number 0 and where each number $x \in \mathbb{R}$ is represented by the point $|x|$ units of distance to the right of 0 (if $x > 0$) or to the left of 0 (if $x < 0$); the line therefore stretches infinitely far in both directions. The set $\mathbb{R}^2$ is represented geometrically as "the Euclidean plane" and the elements of $\mathbb{R}^2$ as points on the plane, where the two components x_1 and x_2 of an ordered pair (a 2-tuple) $(x_1, x_2) \in \mathbb{R}^2$ are the **Cartesian coordinates** of the point associated with the pair (x_1, x_2). We often refer to a pair (x_1, x_2) in $\mathbb{R}^2$ as the *point* (x_1, x_2). Similarly, the set $\mathbb{R}^3$ is represented as a three-dimensional space and the elements (x_1, x_2, x_3) of $\mathbb{R}^3$ as points whose Cartesian coordinates are the components x_1, x_2 and x_3. As in $\mathbb{R}^2$, we often refer to the elements (x_1, x_2, x_3) of $\mathbb{R}^3$ as *points* in $\mathbb{R}^3$. Generalizing this to any natural number n, we refer to $\mathbb{R}^n$ as n-dimensional space and its elements $(x_1, \ldots, x_n)$ as points in $\mathbb{R}^n$, even though for $n > 3$ we can't actually draw a diagram of $\mathbb{R}^n$.

Wherever possible, we'll use boldface letters to denote n-tuples and regular font to denote the components of the n-tuple, like this: $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$. (One reason for the phrase "wherever possible" is that I won't be able to follow this convention on the whiteboard: I haven't found a reasonable way to write boldface letters on the whiteboard.)

We've said that the elements of $\mathbb{R}^n$ are called *points*. But we also call them *vectors*. Here's why, focusing first on $\mathbb{R}^2$. While an ordered pair $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ can be interpreted as a *position* or a *location* — *i.e.*, a *point* — in the $\mathbb{R}^2$-plane, we can also interpret (x_1, x_2) as a *movement* or *displacement* from one location on the plane to another.

Let's consider an example, depicted in Figure 1. Starting from the location $\mathbf{x} = (3, 1)$, suppose we move to the location $\widetilde{\mathbf{x}} = (4, 3)$. This is a movement, or displacement, of $\boldsymbol{\Delta}\mathbf{x} = (1, 2)$. But note that this displacement $\boldsymbol{\Delta}\mathbf{x}$ is itself an ordered pair, an element of $\mathbb{R}^2$. In fact, even the locations $\mathbf{x}$ and $\widetilde{\mathbf{x}}$ can be regarded as displacements from the **origin** of $\mathbb{R}^2$, *i.e.*, from the point $\mathbf{0} = (0, 0)$. So actually, all the elements of $\mathbb{R}^2$ can be interpreted interchangeably as points or as displacements, and we often refer to elements of $\mathbb{R}^2$ generically as **vectors**. Of course, the geometry works the same way in $\mathbb{R}^3$; and generalizing to any n, we continue to use the "vector" terminology in $\mathbb{R}^n$.

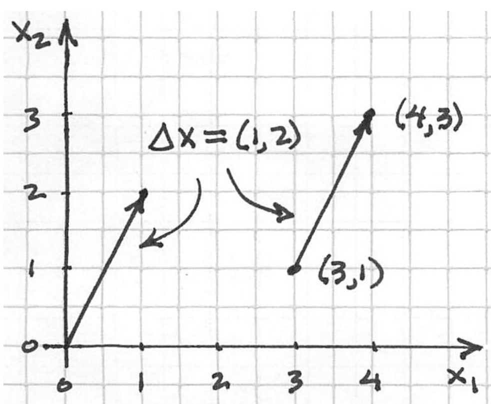

Figure 1

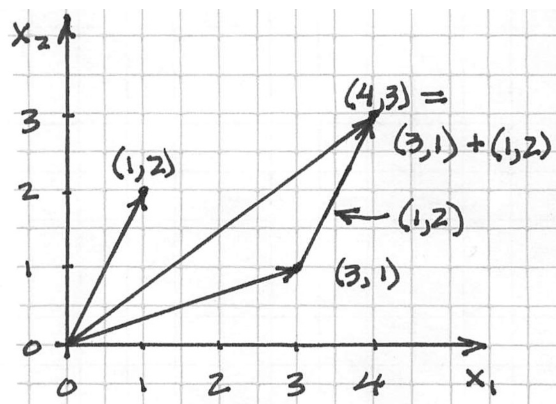

Figure 2

Once we think of $\mathbb{R}^n$ as consisting of vectors, it's natural to also think of adding vectors together. In the example above, starting from the origin $\mathbf{0} = (0, 0)$ in $\mathbb{R}^2$, we can add the two vectors $(3, 1)$ and $(1, 2)$ to give us the vector $(4, 3)$ which is the location (or vector) we end up at, as depicted in Figure 2. It's also natural to think of multiplying a vector by a number — for example, $\frac{1}{2}(4, 3) = (2, 1\frac{1}{2})$, and $-2(1, 2) = (-2, -4)$. We capture the ideas of adding vectors and multiplying them by numbers (called **scalars**) in a formal definition:

Definition: Addition and **scalar multiplication** of n-tuples (vectors) in $\mathbb{R}^n$ are defined component-wise:

$$\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n : \forall \lambda \in \mathbb{R} : \ \ \mathbf{x} + \mathbf{y} := (x_1 + y_1, \ldots, x_n + y_n) \ \text{ and } \ \lambda \mathbf{x} := (\lambda x_1, \ldots, \lambda x_n).$$

Theorem: Addition and scalar multiplication in $\mathbb{R}^n$ have the following properties:

(1) $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n : \mathbf{x} + \mathbf{y} \in \mathbb{R}^n$. ($\mathbb{R}^n$ is **closed under vector addition**.)

(2) $\forall \lambda \in \mathbb{R}, \forall \mathbf{x} \in \mathbb{R}^n : \lambda\mathbf{x} \in \mathbb{R}^n$. ($\mathbb{R}^n$ is **closed under scalar multiplication**.)

(3) $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n : \mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$. (Vector addition is **commutative**.)

(4) $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n : (\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$,
and $\forall \lambda, \mu \in \mathbb{R}, \forall \mathbf{x} \in \mathbb{R}^n : \lambda(\mu\mathbf{x}) = (\lambda\mu)\mathbf{x}$. (Both operations are **associative**.)

(5) $\forall \lambda \in \mathbb{R}, \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n : \lambda(\mathbf{x} + \mathbf{y}) = \lambda\mathbf{x} + \lambda\mathbf{y}$,
and $\forall \lambda, \mu \in \mathbb{R}, \forall \mathbf{x} \in \mathbb{R}^n : (\lambda + \mu)\mathbf{x} = \lambda\mathbf{x} + \mu\mathbf{x}$. (The operations are **distributive**.)

(6) $\forall \mathbf{x} \in \mathbb{R}^n : \mathbf{0} + \mathbf{x} = \mathbf{x}$. ($\mathbf{0}$ is called the **additive identity**.)

(7) $\forall \mathbf{x} \in \mathbb{R}^n : \mathbf{x} + (-\mathbf{x}) = \mathbf{0}$. ($-\mathbf{x}$ is called the **additive inverse** of $\mathbf{x}$.)

(8) $\forall \mathbf{x} \in \mathbb{R}^n : \lambda\mathbf{x} = \mathbf{x}$ for the scalar $\lambda = 1$.

The proofs of the eight properties in the theorem are all elementary, consisting of simply writing the n-tuples component-wise and then applying the definition of addition and/or scalar multiplication.

Length, Distance, and the Dot Product

In an earlier paragraph we added the vector $(3, 1)$ to the vector $(1, 2)$ to give us the vector $(4, 3)$. In Figure 3 we obtain the same vector, $(4, 3)$, as the sum of the vectors $(4, 0)$ and $(0, 3)$. Depicting the sum $(4, 0) + (0, 3)$ this way gives us three vectors that form a right triangle. We can use this to determine the length of the vector $(4, 3)$: assuming the lengths of $(4, 0)$ and $(0, 3)$ are 4 and 3, respectively, the Pythagorean Theorem tells us the *square* of the length of $(4, 3)$ is equal to $4^2 + 3^2$, *i.e.*, 25, so the length of $(4, 3)$ is $\sqrt{25} = 5$. This observation leads us to define the length of any vector $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ as $\sqrt{x_1^2 + x_2^2}$, and to define the length of vectors in $\mathbb{R}^n$ as follows:

Definition: The **length** of a vector $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, denoted $\|\mathbf{x}\|$, and also called the **Euclidean norm** of $\mathbf{x}$, is given by the equation

$$\|\mathbf{x}\| := \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

We call this the *Euclidean* norm because later on we're going to define some other norms. These alternative norms will give us different notions of length and distance in $\mathbb{R}^n$.

Exercise: We appealed to the Pythagorean Theorem to justify, or motivate, the definition of length for vectors: we showed that at least in $\mathbb{R}^2$ our definition does coincide with the usual concept of length. We can do the same thing for $\mathbb{R}^3$ by invoking the Pythagorean Theorem twice in succession: we use $\|\mathbf{x}\|$ to denote the length of a vector $\mathbf{x}$, but we don't use our definition of length; instead we assume only that $\|(x_1, 0, 0)\| = |x_1|$, $\|(0, x_2, 0)\| = |x_2|$, and $\|(0, 0, x_3)\| = |x_3|$; this yields

$$\begin{aligned} \|(x_1, x_2, x_3)\|^2 &= \|(x_1, x_2, 0)\|^2 + \|(0, 0, x_3)\|^2, \text{ by the Pythagorean Theorem} \\ &= \|(x_1, 0, 0)\|^2 + \|(0, x_2, 0)\|^2 + \|(0, 0, x_3)\|^2, \text{ by the Pythagorean Theorem} \\ &= x_1^2 + x_2^2 + x_3^2, \text{ by our assumption about the lengths of these three vectors,} \end{aligned}$$

and therefore $\|(x_1, x_2, x_3)\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$. Draw a diagram in $\mathbb{R}^3$ showing how we can use the Pythagorean Theorem twice, as we've just done for general vectors in $\mathbb{R}^3$, to verify that the length of the vector $(4, 3, 12)$ is 13 — in the same way as Figure 3 showed that the length of $(4, 3)$ is 25.

The following theorem gathers together the fundamental properties of our definition of length. Note how these are all properties we intuitively think the concept of length *should* have.

Theorem: For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and all $\alpha \in \mathbb{R}$:

(N1) $\|\mathbf{x}\| \geqq 0$;

(N2) $\|\mathbf{x}\| = 0 \;\Leftrightarrow\; \mathbf{x} = \mathbf{0}$;

(N3) $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$;

(N4) $\|\mathbf{x} + \mathbf{y}\| \leqq \|\mathbf{x}\| + \|\mathbf{y}\|$, the **triangle inequality**.

Proofs of (N1), (N2), and (N3) are elementary applications of the definition of the norm. We briefly defer the proof of (N4), until we've defined the dot product and proved the Cauchy-Schwarz Inequality.

With a formal definition of the length of vectors, we have an obvious way to define *distance* in $\mathbb{R}^n$: the distance between two points $\mathbf{x}$ and $\widetilde{\mathbf{x}}$ is the length of the displacement vector $\boldsymbol{\Delta}\mathbf{x}$ between $\mathbf{x}$ and $\widetilde{\mathbf{x}}$, where $\boldsymbol{\Delta}\mathbf{x} = \widetilde{\mathbf{x}} - \mathbf{x}$, as depicted in Figure 4.

Definition: The **Euclidean distance** between two vectors $\mathbf{x}$ and $\widetilde{\mathbf{x}}$ in $\mathbb{R}^n$ is the length of the vector $\widetilde{\mathbf{x}} - \mathbf{x}$, *i.e.*, $\|\widetilde{\mathbf{x}} - \mathbf{x}\|$.

Another useful concept in Euclidean space is the dot product, which is closely linked to the concept of length:

Definition: The **dot product** of two vectors $\mathbf{x}$ and $\widetilde{\mathbf{x}}$ in $\mathbb{R}^n$ (also called the **scalar product**, or **inner product**, of $\mathbf{x}$ and $\widetilde{\mathbf{x}}$), denoted $\mathbf{x} \cdot \widetilde{\mathbf{x}}$, is defined as

$$\mathbf{x} \cdot \widetilde{\mathbf{x}} := x_1\widetilde{x}_1 + x_2\widetilde{x}_2 + \cdots + x_n\widetilde{x}_n.$$

Note that for any vector $\mathbf{x} \in \mathbb{R}^n$, we have $\mathbf{x} \cdot \mathbf{x} = \|\mathbf{x}\|^2$, and therefore $\mathbf{x} \cdot \mathbf{x} \geqq 0$ for all $\mathbf{x} \in \mathbb{R}^n$; and we have $\mathbf{x} \cdot \mathbf{x} = 0$ if and only if $\mathbf{x} = \mathbf{0}$.

The following theorem provides some additional important properties of the dot product. Each of the properties follows immediately from the definition.

Theorem: For all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ and all $\alpha \in \mathbb{R}$:

(D1) $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$,

(D2) $\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) = \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z}$,

(D3) $(\alpha\mathbf{x}) \cdot \mathbf{y} = \alpha(\mathbf{x} \cdot \mathbf{y})$,

(D4) $(\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y}) = \mathbf{x} \cdot \mathbf{x} + 2(\mathbf{x} \cdot \mathbf{y}) + \mathbf{y} \cdot \mathbf{y}$.

An important link between the concepts of dot product and length is the Cauchy-Schwarz Inequality (it's also called the Cauchy-Bunyakovsi-Schwarz Inequality). Note that the proof of the inequality uses all four of the properties (D1) - (D4) of the dot product in the preceding theorem.

Theorem (The Cauchy-Schwarz Inequality): For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n : |\mathbf{x} \cdot \mathbf{y}| \leqq \|\mathbf{x}\|\|\mathbf{y}\|$.

Proof: First note that if either $\mathbf{x} = \mathbf{0}$ or $\mathbf{y} = \mathbf{0}$, then $|\mathbf{x} \cdot \mathbf{y}| = \|\mathbf{x}\|\|\mathbf{y}\|$. So we'll assume that both $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{y} \neq \mathbf{0}$.

Define the vector $\alpha\mathbf{x} + \beta\mathbf{y}$, where $\alpha = \|\mathbf{y}\|$ and $\beta = \|\mathbf{x}\|$. We have

$$\begin{aligned}
\|\alpha\mathbf{x} + \beta\mathbf{y}\|^2 &= \big(\|\mathbf{y}\|\mathbf{x} + \|\mathbf{x}\|\mathbf{y}\big) \cdot \big(\|\mathbf{y}\|\mathbf{x} + \|\mathbf{x}\|\mathbf{y}\big) \\
&= \|\mathbf{y}\|^2\mathbf{x} \cdot \mathbf{x} + 2\|\mathbf{x}\|\|\mathbf{y}\|\mathbf{x} \cdot \mathbf{y} + \|\mathbf{x}\|^2\mathbf{y} \cdot \mathbf{y} \\
&= 2\|\mathbf{x}\|^2\|\mathbf{y}\|^2 + 2\|\mathbf{x}\|\|\mathbf{y}\|\mathbf{x} \cdot \mathbf{y} \\
&= 2\|\mathbf{x}\|\|\mathbf{y}\|\big(\|\mathbf{x}\|\|\mathbf{y}\| + \mathbf{x} \cdot \mathbf{y}\big).
\end{aligned}$$

Since $\|\alpha\mathbf{x} + \beta\mathbf{y}\|^2$, $\|\mathbf{x}\|$, and $\|\mathbf{y}\|$ are all non-negative, the above equation yields $\|\mathbf{x}\|\|\mathbf{y}\| + \mathbf{x} \cdot \mathbf{y} \geqq 0$, *i.e.*, $\|\mathbf{x}\|\|\mathbf{y}\| \geqq -\mathbf{x} \cdot \mathbf{y}$.

Now let's consider the vector $\alpha\mathbf{x} + \beta\mathbf{y}$, where $\alpha = \|\mathbf{y}\|$ and $\beta = -\|\mathbf{x}\|$. Now we have

$$\begin{aligned}
\|\alpha\mathbf{x} + \beta\mathbf{y}\|^2 &= \big(\|\mathbf{y}\|\mathbf{x} - \|\mathbf{x}\|\mathbf{y}\big) \cdot \big(\|\mathbf{y}\|\mathbf{x} - \|\mathbf{x}\|\mathbf{y}\big) \\
&= \|\mathbf{y}\|^2\mathbf{x} \cdot \mathbf{x} - 2\|\mathbf{x}\|\|\mathbf{y}\|\mathbf{x} \cdot \mathbf{y} + \|\mathbf{x}\|^2\mathbf{y} \cdot \mathbf{y} \\
&= 2\|\mathbf{x}\|\|\mathbf{y}\|\big(\|\mathbf{x}\|\|\mathbf{y}\| - \mathbf{x} \cdot \mathbf{y}\big),
\end{aligned}$$

so by the same argument as above we now have $\|\mathbf{x}\|\|\mathbf{y}\| - \mathbf{x} \cdot \mathbf{y} \geqq 0$, *i.e.*, $\|\mathbf{x}\|\|\mathbf{y}\| \geqq \mathbf{x} \cdot \mathbf{y}$.

We've now established that $\|\mathbf{x}\|\|\mathbf{y}\| \geqq -\mathbf{x} \cdot \mathbf{y}$ and $\|\mathbf{x}\|\|\mathbf{y}\| \geqq \mathbf{x} \cdot \mathbf{y}$ — *i.e.*,

$$\|\mathbf{x}\|\|\mathbf{y}\| \geqq \max\{\mathbf{x} \cdot \mathbf{y}, -\mathbf{x} \cdot \mathbf{y}\} = |\mathbf{x} \cdot \mathbf{y}|,$$

completing the proof of the Cauchy-Schwarz Inequality. ■

Remark: The Cauchy-Schwarz Inequality is an equation — *i.e.*, $|\mathbf{x} \cdot \mathbf{y}| = \|\mathbf{x}\|\|\mathbf{y}\|$ — if and only if one of the vectors $\mathbf{x}$ or $\mathbf{y}$ is a scalar multiple of the other.

Proof: The equation obviously holds if either $\mathbf{x} = \mathbf{0}$ or $\mathbf{y} = \mathbf{0}$, so we'll assume that both $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{y} \neq \mathbf{0}$. First note that if $|\mathbf{x} \cdot \mathbf{y}| = \|\mathbf{x}\|\|\mathbf{y}\|$, then either $\|\mathbf{x}\|\|\mathbf{y}\| - \mathbf{x} \cdot \mathbf{y} = 0$ or $\|\mathbf{x}\|\|\mathbf{y}\| + \mathbf{x} \cdot \mathbf{y} = 0$. In either case this implies, according to the equations in the proof of the Cauchy-Schwarz Inequality, that $\|\alpha\mathbf{x} + \beta\mathbf{y}\|^2 = 0$ — *i.e.*, that $\alpha\mathbf{x} + \beta\mathbf{y} = \mathbf{0}$, which implies that $\mathbf{x}$ and $\mathbf{y}$ are scalar multiples of one another. Conversely, suppose $\mathbf{y}$ is a scalar multiple of $\mathbf{x}$ — *i.e.*, $\mathbf{y} = t\mathbf{x}$ for some $t \in \mathbb{R}$. Then

$$|\mathbf{x} \cdot \mathbf{y}| = |\mathbf{x} \cdot (t\mathbf{x})| = |t(\mathbf{x} \cdot \mathbf{x})| = |t|(\mathbf{x} \cdot \mathbf{x}) = |t|\|\mathbf{x}\|^2 = |t|\|\mathbf{x}\|\|\mathbf{x}\| = \|\mathbf{x}\|\|t\mathbf{x}\| = \|\mathbf{x}\|\|\mathbf{y}\|. \ ■$$

Now we can return to the Triangle Inequality and provide a simple proof that uses the Cauchy-Schwarz Inequality:

Theorem (The Triangle Inequality): For all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$: $\|\mathbf{x}+\mathbf{y}\| \leqq \|\mathbf{x}\| + \|\mathbf{y}\|$.

Proof:

$$\begin{aligned} \|\mathbf{x}+\mathbf{y}\|^2 &= \|\mathbf{x}\|^2 + 2\mathbf{x}\cdot\mathbf{y} + \|\mathbf{y}\|^2 \\ &\leqq \|\mathbf{x}\|^2 + 2|\mathbf{x}\cdot\mathbf{y}| + \|\mathbf{y}\|^2 \\ &\leqq \|\mathbf{x}\|^2 + 2\|\mathbf{x}\|\|\mathbf{y}\| + \|\mathbf{y}\|^2 \\ &= \left(\|\mathbf{x}\| + \|\mathbf{y}\|\right)^2, \end{aligned}$$

which, since $\|\mathbf{x}\| + \|\mathbf{y}\| \geqq 0$, yields $\|\mathbf{x}+\mathbf{y}\| \leqq \|\mathbf{x}\| + \|\mathbf{y}\|$. ■

The following remark provides a necessary and sufficient condition for the Triangle Inequality to be an equation:

Remark: $\|\mathbf{x}+\mathbf{y}\| = \|\mathbf{x}\| + \|\mathbf{y}\|$ if and only if one of the vectors $\mathbf{x}$ or $\mathbf{y}$ is a non-negative scalar multiple of the other.

Proof: The equation obviously holds if either vector is $\mathbf{0}$, so we assume that both $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{y} \neq \mathbf{0}$. In the proof of the Triangle Inequality it's clear that $\|\mathbf{x}+\mathbf{y}\| = \|\mathbf{x}\| + \|\mathbf{y}\|$ if and only if both inequalities are equations — *i.e.*, if and only if $\mathbf{x}\cdot\mathbf{y} = |\mathbf{x}\cdot\mathbf{y}|$ and $|\mathbf{x}\cdot\mathbf{y}| = \|\mathbf{x}\|\|\mathbf{y}\|$. The second equation is true if and only if $\mathbf{y} = t\mathbf{x}$ for some $t \in \mathbb{R}$, as we've shown above; and given that $\mathbf{y} = t\mathbf{x}$, the first equation is true if and only if $t \geqq 0$. ■

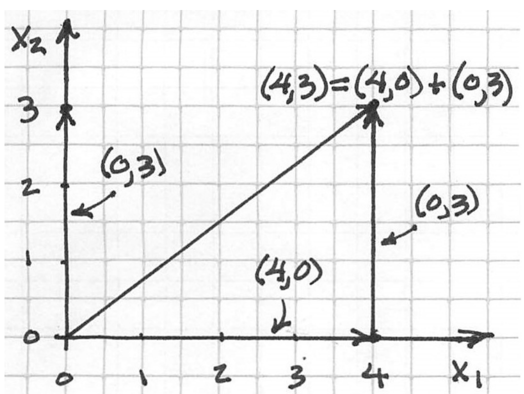

Figure 3

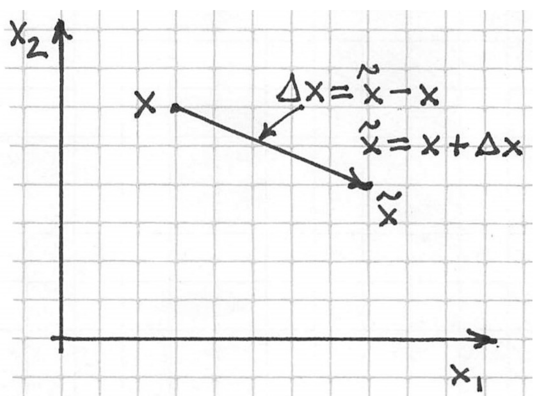

Figure 4