Notation for Sets of Functions and Subsets

Recall that we defined a sequence as a function on the domain $\mathbb{N}$ of natural numbers. Thus, a sequence of real numbers — *i.e.*, an element $\mathbf{x} = (x_1, x_2, \ldots) \in \mathbb{R}^\infty$ — is a function $\mathbf{x} : \mathbb{N} \to \mathbb{R}$. A sequence of elements of a set X is a function $\mathbf{x} : \mathbb{N} \to X$, and we would denote the set of all such sequences as X^∞.

We can do the same for "finite sequences":

An ordered pair of real numbers, $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, is a function $\mathbf{x} : \{1, 2\} \to \mathbb{R}$.

An n-tuple $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is a function $\mathbf{x} : \{1, 2, \ldots, n\} \to \mathbb{R}$.

In each case, the argument of the function can be written either as a subscript or, in the notation that's more common for a function, inside parentheses — either x_i or $x(i)$, it's really just a matter of convenience (and convention!) which way we choose to do it.

And note that we write

$\mathbb{R}^2$ for the set of all functions from $\{1, 2\}$ into $\mathbb{R}$;

$\mathbb{R}^n$ for the set of all functions from $\{1, 2, \ldots, n\}$ into $\mathbb{R}$;

$\mathbb{R}^\infty$ for the set of all functions from $\{1, 2, \ldots\}$ into $\mathbb{R}$.

Now let X and Y be arbitrary sets. The notation we use for the set of all functions $f : X \to Y$ is the following:

Notation: For any sets X and Y, the set of all functions $f : X \to Y$ is denoted Y^X .

Here's another useful piece of standard notation:

Notation: For finite sets X, the number of elements of X is denoted $|X|$, or sometimes $\#X$.

Exercise: Let $X = \{a, b, c\}$ and $Y = \{\textit{red, green}\} = \{r, g\}$. Enumerate all the elements of the set Y^X — *i.e.*, all the functions that assign to each $x \in X$ either the color red or the color green. You should find that there are eight functions — *i.e.*, $|Y^X| = 8$. Note that $|Y|^{|X|} = 2^3 = 8$ as well.

Exercise: Let X and Y be finite sets — say, $|X| = n$ and $|Y| = m$. Without loss of generality, you could let $X = \{1, \ldots, n\}$ and $Y = \{1, \ldots, m\}$. Verify that $|Y^X| = |Y|^{|X|}$ — *i.e.*, $|Y^X| = m^n$.

Exercise: Let $Y = \{0, 1\}$ and let X be any finite set. Without loss of generality, you could let $X = \{1, \ldots, n\}$. Verify that the number of distinct subsets of X is 2^n — *i.e.*, it is $2^{|X|}$, or $|\{0, 1\}|^{|X|}$, which is $|\{0, 1\}^X|$. But $|\{0, 1\}^X|$ is also the number of functions from X into $\{0, 1\}$, because $\{0, 1\}^X$ is the set of all functions from X into $\{0, 1\}$.

This last exercise motivates the idea of indicator functions:

Definition: Let X be a set. For each subset $S \subseteq X$, define the **indicator function** I_S of the set S as follows:

$$I_S(x) = \begin{cases} 0\,, & \text{if } x \notin S \\ 1\,, & \text{if } x \in S. \end{cases}$$

Thus, the set of all indicator functions on X is essentially the same as the set of all subsets of X: every function from X into $\{0, 1\}$ corresponds to a distinct subset of X, and every subset of X corresponds to a distinct function from X into $\{0, 1\}$. We could therefore use the notation $\{0, 1\}^X$ — the set of all functions from X into $\{0, 1\}$ — for the set of all subsets of X. However, the convention is instead to just write 2^X for the set of all subsets of X, meaning the set of all functions from X into a two-point set, the set of all indicator functions on X.

Notation: For any set X, the set of all subsets of X is denoted 2^X, sometimes called the **power set** of X.