Convergence of Sequences of Functions: Some Additional Notes

Let X be an arbitrary set. We won't assume any algebraic (*i.e.*, vector space) or metric structure for X, except in some of the examples (where, for example, X may be a subset of $\mathbb{R}$, with the usual properties it inherits from $\mathbb{R}$).

Definition: A sequence $\{f_n\}$ of functions $f_n : X \to \mathbb{R}$ **converges pointwise** to a function $f : X \to \mathbb{R}$ if

$$\forall x \in X \colon \forall \epsilon > 0 \colon \exists \bar{n} \in \mathbb{N} \colon n > \bar{n} \Rightarrow |f_n(x) - f(x)| < \epsilon.$$

In other words, $\{f_n\}$ converges pointwise to f if for every point $x \in X$, the sequence of real numbers $\{f_n(x)\}$ converges to the number $f(x)$.

Note that by reversing the order of the universal quantifiers $\forall x \in X$ and $\forall \epsilon > 0$ in the condition above, defining pointwise convergence, the condition can be equivalently written as

$$(*) \qquad \forall \epsilon > 0 \colon \forall x \in X \colon \exists \bar{n} \in \mathbb{N} \colon n > \bar{n} \Rightarrow |f_n(x) - f(x)| < \epsilon.$$

We're going to be interested in *bounded* functions, and in the convergence of sequences of bounded functions. We'll denote the set of all bounded real-valued functions $f : X \to \mathbb{R}$ on a set X as $B(X)$. If the set $B(X)$ is endowed with a metric d, then $(B(X), d)$ is a metric space, and our definition of convergence of sequences in a metric space then applies to $(B(X), d)$:

A sequence $\{f_n\}$ converges to f in the metric space $(B(X), d)$ if

$$\forall \epsilon > 0 \colon \exists \bar{n} \in \mathbb{N} \colon n > \bar{n} \Rightarrow d(f_n, f) < \epsilon.$$

We're going to be working with bounded real-valued functions, so the sup-norm is well-defined (because for any function $f \in B(X)$ the set $f(X)$ is a bounded set in $\mathbb{R}$, and therefore the Monotone Convergence Theorem applies), so we can use the sup-norm as our metric in $B(X)$:

$$d(f, g) = \|f - g\|_\infty = \sup\{|f(x) - g(x)| \mid x \in X\}.$$

So our definition of convergence in the metric space $(B(X), d)$ becomes

$$\{f_n\} \text{ converges to } f \text{ if and only if } \forall \epsilon > 0 \colon \exists \bar{n} \in \mathbb{N} \colon n > \bar{n} \Rightarrow \|f_n - f\|_\infty < \epsilon.$$

Example 1: Let $X = \{a, b, c\}$; let $f : X \to \mathbb{R}$ be defined by $\forall x \in X : f(x) = 0$; and for each $n \in \mathbb{N}$ let $f_n(a) = (-1)^n \frac{1}{n}$, $f_n(b) = (-1)^{n+1} \frac{1}{n^2}$, and $f_n(c) = (-1)^{2n} \frac{1}{n^3}$. Then $\{f_n\}$ obviously converges pointwise to f. Also, $\{f_n\}$ converges to f in $(B(X), \|\cdot\|_\infty)$, because for each $n \in \mathbb{N}$, $|f_n(a)| > |f_n(b)| > |f_n(c)|$, so $\|f_n - f\|_\infty = |f_n(a)| = \frac{1}{n}$.

It's straightforward to show that Example 1 generalizes to all finite sets X:

Proposition 1: If X is a finite set, then a sequence $\{f_n\}$ of functions $f_n : X \to \mathbb{R}$ converges pointwise to f if and only if $\{f_n\}$ converges to f in the metric space $(B(X), \|\cdot\|_\infty)$.

(Note that if X is finite, then every function $f : X \to \mathbb{R}$ is bounded — *i.e.*, is in $B(X)$.)

Example 2: Let f and the sequence $\{f_n\}$ of functions in $B([0,1])$ be defined as follows:

$$f_n(x) = \begin{cases} 1, & \text{if } x \leqq \frac{1}{n} \\ 0\,, & \text{if } x > \frac{1}{n} \end{cases} \qquad\qquad f(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0\,, & \text{if } x > 0. \end{cases}$$

Then

(a) $\{f_n\}$ converges pointwise to f:

If $x = 0$: $f_n(x) = 1 = f(x)$ for all $n \in \mathbb{N}$.
If $x > 0$: Let $\bar{n}$ be any $\bar{n} \in \mathbb{N}$ such that $\bar{n} > \frac{1}{x}$ — *i.e.*, $x > \frac{1}{\bar{n}}$. Then for any $n \geqq \bar{n}$ we have $x > \frac{1}{\bar{n}} > \frac{1}{n}$ and therefore $f_n(x) = 0 = f(x)$.

(b) $\{f_n\}$ does not converge to f in $B([0,1], \|\cdot\|_\infty)$:

If $0 < x < \frac{1}{n}$ then $f_n(x) = 1$, while $f(x) = 0$; therefore $\|f_n - f\|_\infty = 1$ for all $n \in \mathbb{N}$.

It might seem that the convergence failure in (b) in Example 2 is because the functions f_n are discontinuous. The following example shows that this is not the case: the functions f_n in this example are all continuous (and they again converge pointwise), but they don't converge in the sup-norm.

Example 3: For each $n \in \mathbb{N}$ let $f_n : [0,1] \to \mathbb{R}$ be the function

$$f_n(x) = \begin{cases} nx, & \text{if } x \leqq \frac{1}{n} \\ 1\,, & \text{if } x \geqq \frac{1}{n}. \end{cases}$$

Exercise: Verify in the preceding example that $\{f_n\}$ converges pointwise but not in the metric space $(B(X), \|\cdot\|_\infty)$.

While the preceding examples show that pointwise convergence does not imply convergence in the sup-norm, they suggest that perhaps sup-norm convergence does imply pointwise convergence. The following proposition verifies that this conjecture is correct. The proof is left as an exercise.

Proposition 2: If $\{f_n\}$ converges to f in the metric space $(B(X), \|\cdot\|_\infty)$, then $\{f_n\}$ converges pointwise to f.

The reason we can't establish sup-norm convergence in the examples is because this kind of convergence requires that for any ϵ we can find an $\bar{n}$ for which $n > \bar{n}$ implies that $\|f_n - f\|_\infty < \epsilon$, and therefore that $|f_n(x) - f(x)| < \epsilon$ *for every* $x \in X$. The $\bar{n}$ depends on ϵ, but it can't depend on x — unlike with pointwise convergence, where the $\bar{n}$ can depend on ϵ differently for different values of x. Convergence in the sup-norm is convergence that's *uniform* across all of X.

Definition: A sequence $\{f_n\}$ of functions $f_n : X \to \mathbb{R}$ **converges uniformly** to a function $f : X \to \mathbb{R}$ if

$$(**) \qquad \forall \epsilon > 0 \colon \exists \bar{n} \in \mathbb{N} \colon \forall x \in X : n > \bar{n} \Rightarrow |f_n(x) - f(x)| < \epsilon.$$

Notice that the only difference between $(*)$ and $(**)$ is the placement of the universal quantifier "$\forall x \in X :$". Here it comes after the existence of $\bar{n}$, so the single $\bar{n}$ has to work for all $x \in X$; for pointwise convergence it comes before the existence of $\bar{n}$, so different numbers $\bar{n}$ can be used for different values of x.

The following proposition establishes that uniform convergence is really just another name for sup-norm convergence:

Proposition 3: A sequence $\{f_n\}$ in $B(X)$ converges uniformly to a function $f \in B(X)$ if and only if it converges in the metric $\|\cdot\|_\infty$ on $B(X)$.

Proof: Let $\epsilon > 0$.

(a) If $\|f_n - f\|_\infty < \epsilon$ then $\forall x \in X : |f_n(x) - f(x)| < \epsilon$, from which it follows from the definitions that if $\{f_n\}$ converges to f in $\|\cdot\|_\infty$ then it converges uniformly to f.

(b) If $\forall x \in X : |f_n(x) - f(x)| < .99\,\epsilon$, for example, then $\|f_n - f\|_\infty < \epsilon$, from which it follows from the definitions that if $\{f_n\}$ converges uniformly to f then it converges to f in $\|\cdot\|_\infty$. ■

Uniform convergence really is a stronger kind of convergence than merely pointwise convergence for sequences of functions. The following exercise demonstrates that pointwise convergence is not enough to ensure uniform convergence even in the best of cases, where both the domain and target space are the unit interval in $\mathbb{R}$ and all the functions (including the limit function) are continuous — unlike Examples 2 and 3, in each of which the sequence of functions merely converged pointwise but not uniformly, and the limit function was not continuous.

Exercise: Provide a counterexample to show that the following conjecture is false: Let f be a bounded continuous real-valued function on $[0, 1] \subseteq \mathbb{R}$, and for each $n \in \mathbb{N}$, let f_n be a bounded continuous real-valued function on $[0, 1]$. If $\{f_n\}$ converges pointwise to f then $\{f_n\}$ converges uniformly to f.

The following theorem shows that if a sequence of continuous functions *does* converge uniformly, then the limit function will be continuous.

Theorem: Let (X, d) be a metric space, and for each $n \in \mathbb{N}$, let $f_n : X \to \mathbb{R}$ be a continuous real-valued function on X. If $\{f_n\}$ converges uniformly to $f : X \to \mathbb{R}$, then f is continuous.

Proof: Let $\overline{x} \in X$ and let $\epsilon > 0$. Because $\{f_n\}$ converges uniformly to f, there is an $\overline{n} \in \mathbb{N}$ such that

$$n > \overline{n} \Rightarrow \forall x \in X : |f_n(x) - f(x)| < \frac{\epsilon}{3}.$$

Let n be any integer greater than $\overline{n}$. Since f_n is continuous, there is a $\delta > 0$ such that

$$d(x, \overline{x}) < \delta \Rightarrow |f_n(x) - f_n(\overline{x})| < \frac{\epsilon}{3}.$$

For any x that satisfies $d(x, \overline{x}) < \delta$, the Triangle Inequality therefore yields

$$|f(x) - f(\overline{x})| < |f(x) - f_n(x)| + |f_n(x) - f_n(\overline{x})| + |f_n(\overline{x}) - f(\overline{x})| < \epsilon. \quad \blacksquare$$