Correspondences

Functions are single-valued. For a function $f : X \to Y$, every $x \in X$ is mapped to one *and only one* point $y \in Y$, the point $y = f(x)$. Thus, for example, when a consumer with a strictly quasiconcave utility function behaves according to the Utility Maximization Hypothesis, we can summarize his market behavior by his demand function: the bundle he chooses is a (single-valued) function of the price-list he faces.

But suppose the consumer's utility function is not strictly quasiconcave; for example, suppose it is $u(x_1, x_2) = ax_1 + x_2$. For any price-list that satisfies $p_1 = ap_2$, this consumer is indifferent among *all* the bundles on his budget constraint $\{\mathbf{x} \in \mathbb{R}^2_+ \mid p_1x_1 + p_2x_2 = p_1\mathring{x}_1 + p_2\mathring{x}_2\} = \{\mathbf{x} \in \mathbb{R}^2_+ \mid ax_1 + x_2 = a\mathring{x}_1 + \mathring{x}_2\}$. He'll choose *some* bundle on his budget constraint, but we can't say which bundle it will be.

This situation, in which we need to analyze behavior that does not manifest itself in uniquely determined actions, is extremely common in economics and game theory. And if individual behavior is not single-valued, then aggregate behavior won't be single-valued either: in the demand theory example above, if some consumer's demand function is not single-valued, then the market demand function won't be single-valued. We evidently need a new analytical tool, the *multivalued function* or *correspondence*.

Definition: A **correspondence** f from a set X to a set Y, denoted $f : X \twoheadrightarrow Y$, is a function from X to the set 2^Y of all subsets of Y. Correspondences are also called multivalued functions or set-valued functions.

A correspondence $f : X \twoheadrightarrow Y$ is therefore a set-valued function from X to Y — for every $x \in X$, $f(x)$ is a subset of Y.

The following examples are depicted in Figures 1-3.

Example 1: $f : \mathbb{R}_+ \twoheadrightarrow \mathbb{R}$ is defined by $f(x) = \{\sqrt{x}, -\sqrt{x}\}$.

Example 2: $f : \mathbb{R}_+ \twoheadrightarrow \mathbb{R}$ is defined by

$$f(x) = \begin{cases} [0, \frac{1}{x}] & \text{if } x > 0 \\ \{0\} & \text{if } x = 0. \end{cases}$$

Example 3: $f : \mathbb{R}_+ \twoheadrightarrow [0,1]$ is defined by

$$f(x) = \begin{cases} [x, \frac{1}{2}] & \text{if } 0 \leqq x < \frac{1}{2} \\ \{\frac{1}{4}, \frac{1}{2}\} \cup [\frac{3}{4}, 1] & \text{if } x = \frac{1}{2} \\ \{\frac{1}{4}\} \cup [\frac{3}{4}, 1] & \text{if } x > \frac{1}{2} \end{cases}$$

Definition: The **graph** of a correspondence $f : X \twoheadrightarrow Y$, denoted $\mathrm{Gr}(f)$, is the set

$$\mathrm{Gr}(f) := \{(x,y) \in X \times Y \mid y \in f(x)\ \}.$$

Definition: If $f : X \twoheadrightarrow Y$ is a correspondence, a function $\mathring{f} : X \to Y$ is a **selection** from f if $\forall x \in X : \mathring{f}(x) \in f(x)$ — equivalently, $\forall x \in X : \{\mathring{f}(x)\} \subseteq f(x)$.

Remark: A correspondence $f : X \twoheadrightarrow Y$ has a unique selection $\mathring{f}$ if and only if f is everywhere singleton-valued — *i.e.*, $\forall x \in X : f(x)$ is a singleton, *i.e.*, $\forall x \in X : f(x) = \{\mathring{f}(x)\}$.

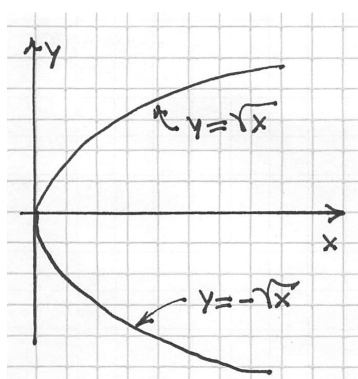

Figure 1

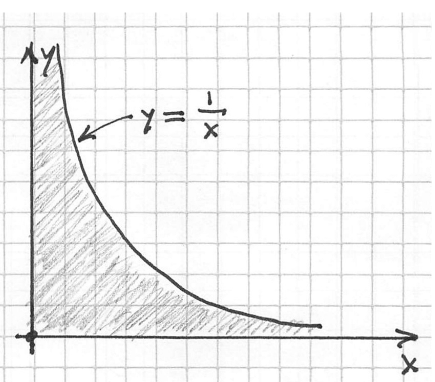

Figure 2

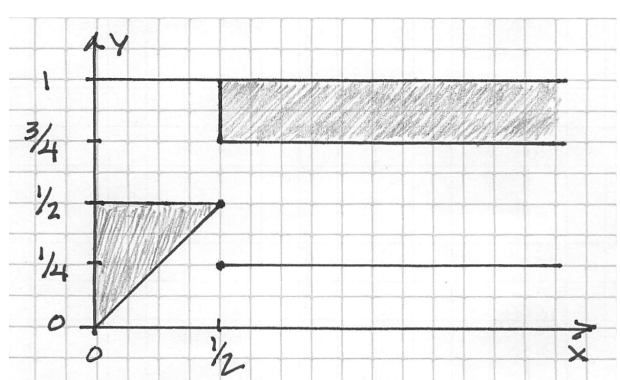

Figure 3

Continuity of Correspondences

From now on we'll assume that (X, d_X) and (Y, d_Y) are metric spaces. In order to develop the idea of continuity for correspondences, let's begin by recalling the definition of a continuous function:

Definition: A function $f : X \to Y$ is **continuous** at $\overline{x}$ if for every open set V such that $f(\overline{x}) \in V$, there is an open set U such that $\overline{x} \in U$ and $x \in U \Rightarrow f(x) \in V$.

Can we adapt this definition to correspondences in a way that still captures the idea of continuity? At first it may appear that we can't: if f is a correspondence instead of a function, then "$f(\overline{x}) \in V$" and "$f(x) \in V$" in the definition have no meaning, because $f(\overline{x})$ and $f(x)$ are sets. But notice that, for functions at least, we can write "$f(x) \in V$" in two different but equivalent ways:

$$\text{(a) } \{f(x)\} \subseteq V \quad \text{and} \quad \text{(b) } \{f(x)\} \cap V \neq \emptyset,$$

both of which are equivalent to $f(x) \in V$. So when f is a correspondence, we'll replace "$f(x) \in V$" with either

$$\text{(a}'\text{) } f(x) \subseteq V \quad \text{or} \quad \text{(b}'\text{) } f(x) \cap V \neq \emptyset.$$

This will give us two alternative definitions — note that (a′) and (b′) are different if $f(x)$ is not a singleton — and under both definitions a singleton-valued correspondence f will be "continuous" if and only if its unique selection $\mathring{f}(x)$ is a continuous function.

Definition: A correspondence $f : X \twoheadrightarrow Y$ is

(a) **upper hemicontinuous** (UHC) at $\overline{x} \in X$ if for every open set V such that $f(\overline{x}) \subseteq V$, there is an open set U such that $\overline{x} \in U$ and $x \in U \Rightarrow f(x) \subseteq V$;

(b) **lower hemicontinuous** (LHC) at $\overline{x} \in X$ if for every open set V such that $f(\overline{x}) \cap V \neq \emptyset$, there is an open set U such that $\overline{x} \in U$ and $x \in U \Rightarrow f(x) \cap V \neq \emptyset$.

(c) **continuous** at $\overline{x} \in X$ if it is both UHC and LHC at $\overline{x}$.

(d) A correspondence is UHC/LHC/continuous on X if it is UHC/LHC/continuous at each $\overline{x} \in X$.

Exercise: Verify that in Example 1 the correspondence f is both UHC and LHC on $\mathbb{R}_+$. Verify that in Example 2, f is both UHC and LHC at every $\overline{x} \in \mathbb{R}_{++}$, and that f is LHC but not UHC at $\overline{x} = 0$. Verify that in Example 3, f is UHC but not LHC at $\overline{x} = \frac{1}{2}$.

Here is another continuity property of correspondences that's important:

Definition: A correspondence $f : X \twoheadrightarrow Y$ is **closed** if it has a **closed graph**, *i.e.*, if $\mathrm{Gr}(f)$ is a closed subset of $X \times Y$.

Remark: A correspondence $f : X \twoheadrightarrow Y$ has a closed graph if and only if it satisfies the following condition: if $\{x_n\}$ and $\{y_n\}$ are sequences in X and Y such that $y_n \in f(x_n)$ for each n, and $\{x_n\} \to x$ and $\{y_n\} \to y$, then $y \in f(x)$.

Proof: The condition can be stated as follows: if $\{(x_n, y_n)\}$ is a sequence in $X \times Y$ such that $(x_n, y_n) \in Gr(f)$ for each n and $\{(x_n, y_n)\} \to (x, y)$, then $(x, y) \in Gr(f)$. $\|$

The closed graph property is generally more intuitive than the UHC property, and therefore it's usually easier to work with. Moreover, in many applications the target space Y is compact, in which case any closed correspondence is UHC, as we'll show momentarily. Note that Examples 1 and 3 both have closed graphs and are UHC. Example 2 is not UHC and its graph is not closed.

Theorem: If Y is compact, then any correspondence $f : X \twoheadrightarrow Y$ that has a closed graph is UHC on X.

Proof: We assume that f has a closed graph but is not UHC at some $\overline{x} \in X$, and we will obtain a contradiction. Since f is not UHC at $\overline{x}$, there is an open set $V \subseteq Y$ such that $f(\overline{x}) \subseteq V$ but for every open ball of the form $B(\overline{x}, 1/n)$ there is an $x_n \in B(\overline{x}, 1/n)$ such that $f(x_n) \nsubseteq V$ — *i.e.*, such that some $y_n \in f(x_n)$ satisfies $y_n \notin V$.

Since Y is compact, $\{y_n\}$ has a convergent subsequence, which we also write as $\{y_n\}$, and we write $\overline{y} = \lim y_n$. Since f has a closed graph, we have $\overline{y} \in f(\overline{x})$. Since $\overline{y} \in f(\overline{x}) \subseteq V$ and V is open, there is an $\epsilon > 0$ such that $B(\overline{y}, \epsilon) \subseteq V$. Since $y_n \notin V$ for all n, we have $y_n \notin B(\overline{y}, \epsilon)$ for all n. But then $\{y_n\} \not\to \overline{y}$, a contradiction. $\|$

Example 4: $f : [0, 1] \twoheadrightarrow [0, 1]$ is defined by

$$f(x) = \begin{cases} [.3, .7] & \text{if } x \leqq \frac{1}{2} \\ \{\frac{1}{2}\} & \text{if } x > \frac{1}{2}. \end{cases}$$

See Figure 4. This correspondence has a closed graph (and is therefore UHC, since the target space is compact), but it's not LHC at $x = \frac{1}{2}$. .

Example 5: $f : \mathbb{R}_+ \twoheadrightarrow \mathbb{R}$ is defined by

$$f(x) = \begin{cases} \{\frac{1}{x}\} & \text{if } x > 0 \\ \{0\} & \text{if } x = 0. \end{cases}$$

See Figure 5. This correspondence is singleton-valued, and its unique selection is discontinuous at $\overline{x} = 0$, so it's neither UHC nor LHC at $\overline{x} = 0$. (You should be able to prove this directly from the definitions of UHC and LHC.) However, it does have a closed graph. But since its target space is $\mathbb{R}_+$, which is not compact, this is not inconsistent with the above theorem. Instead, the example is a counterexample to demonstrate that compactness cannot be dispensed with in the theorem: if Y is not compact, a correspondence with a closed graph may not be UHC.

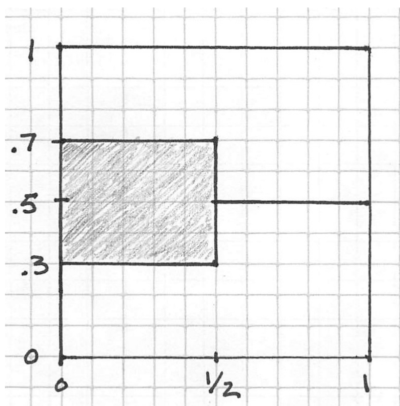

Figure 4

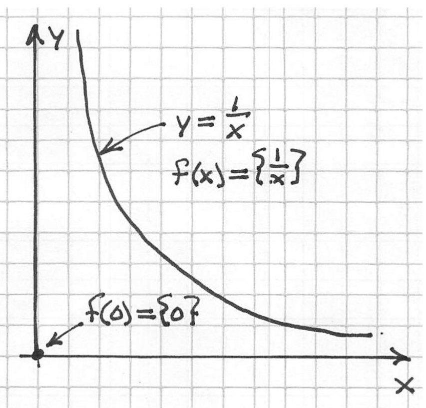

Figure 5

Exercise: Verify the claims made above about the examples' properties.

The following example demonstrates that the converse of the above theorem is false: even if Y is compact, a UHC correspondence need not have a closed graph.

Example 6: $f : [0,1] \twoheadrightarrow [0,1]$ is defined by $f(x) = (.3, .7)$ for all $x \in [0,1]$.

This correspondence is continuous (both UHC and LHC), as every constant correspondence is, and its target space is compact, but it does not have a closed graph.

Exercise: Verify that every constant correspondence is continuous.

Just as the closed graph property is useful because it can be characterized in terms of convergent sequences, there is a similar convergent-sequence characterization of lower hemicontinuity that is often easier to use than the definition of LHC.

Theorem: A correspondence $f : X \twoheadrightarrow Y$ is LHC at $x \in X$ if and only if for every sequence $\{x_n\} \to x$ and every $y \in f(x)$, there is a sequence $\{y_n\}$ that satisfies both (1) $y_n \in f(x_n)$ for all n and (2) $\{y_n\} \to y$.

Proof: de la Fuente, p. 111.

Here are several useful facts about correspondences. The proofs are straightforward and are good exercises for understanding correspondences and their continuity properties.

Theorem: If Y is compact and the correspondences $f : X \twoheadrightarrow Y$ and $g : X \twoheadrightarrow Y$ both have closed graphs, then the sum $f + g$ also has a closed graph, where $f + g$ is the correspondence defined by

$$(f+g)(x) := \{ \, y_1 + y_2 \in Y \mid y_1 \in f(x) \quad \text{and} \quad y_2 \in g(x) \, \}.$$

Theorem: If Y and Z are compact and the correspondences $f : X \twoheadrightarrow Y$ and $g : Y \twoheadrightarrow Z$ both have closed graphs, then the composition $f \circ g$ also has a closed graph, where $f \circ g$ is the correspondence defined by

$$\begin{aligned}(f \circ g)(x) := g(f(x)) &= \{ \, z \in Z \mid \exists y \in Y : y \in f(x) \ \& \ z \in g(y) \, \} \\ &= \bigcup \{ \, g(y) \mid y \in f(x) \, \}.\end{aligned}$$

Theorem: If the correspondences $f : X \twoheadrightarrow Y$ and $g : X \twoheadrightarrow Y$ are both UHC and have closed graphs, then the correspondence $f \cap g : X \twoheadrightarrow Y$ is UHC, where $f \cap g$ is defined by $(f \cap g)(x) := f(x) \cap g(x)$.

Reference: The book *Fixed Point Theorems with Applications to Economics and Game Theory*, by Kim Border, contains a detailed account of the continuity properties of correspondences (in Chapter 11), as well as the Maximum Theorem (in Chapter 12). The Maximum Theorem will be introduced below.

Fixed Points of Correspondences

Definition: A fixed point of a correspondence $f : X \twoheadrightarrow X$ is an $x^* \in X$ for which $x^* \in f(x^*)$.

The most commonly used generalization of Brouwer's Theorem to correspondences is Kakutani's Fixed Point Theorem.

Kakutani's Theorem: Let $f : S \twoheadrightarrow S$ be a correspondence. If S is nonempty, compact, and convex, and if f is nonempty-valued, convex-valued, and has a closed graph, then f has a fixed point.

When we say that f is nonempty-valued and convex-valued, we mean that $f(x)$ is a nonempty convex set for every $x \in S$.

Example 7: $f : [0,1] \twoheadrightarrow [0,1]$ is defined by

$$f(x) = \begin{cases} [.6, .8] & \text{if } x < \frac{1}{2} \\ [.2, .8] & \text{if } x = \frac{1}{2} \\ [.2, .4] & \text{if } x > \frac{1}{2}. \end{cases}$$

This correspondence is convex-valued and has a closed graph, and it therefore has a fixed point. Its unique fixed point is $x^* = \frac{1}{2}$.

Example 8: $f : [0,1] \twoheadrightarrow [0,1]$ is defined by

$$f(x) = \begin{cases} \{.7\} & \text{if } x < \frac{1}{2} \\ [.2, .4] & \text{if } x \geqq \frac{1}{2}. \end{cases}$$

This correspondence has no fixed point. The correspondence is convex-valued but does not have a closed graph: it's discontinuous at $x = \frac{1}{2}$.

Example 9: $f : [0,1] \twoheadrightarrow [0,1]$ is defined by

$$f(x) = \begin{cases} [.6, .8] & \text{if } x < \frac{1}{2} \\ [.2, .4] \cup [.6, .8] & \text{if } x = \frac{1}{2} \\ [.2, .4] & \text{if } x > \frac{1}{2}. \end{cases}$$

This correspondence has no fixed point. The correspondence has a closed graph but is not convex-valued: it's convex-valued at every x except $x = \frac{1}{2}$.

The Maximum Theorem

Economic models of individual behavior generally assume optimizing behavior. We typically use the classical Weierstrass Theorem ("a continuous real-valued function on a compact set attains a maximum") to infer that there is actually an optimizing action available for the individual to choose. Is the individual's behavior continuous? That is, does the individual's chosen action respond continuously to changes in his environment? When we have a specific functional form for the objective function (*e.g.*, a Cobb-Douglas utility function), we can often obtain a closed-form expression for the behavioral function and determine directly whether it's continuous. Even when we can't obtain a closed-form behavioral function, if the objective function is differentiable we can generally apply the Implicit Function Theorem to establish continuity (and even differentiability) of the behavioral function.

The preceding paragraph implicitly assumed that the individual's behavior is described by a *single-valued* function — that the optimizing action is always unique. We now have correspondences at our disposal, so we can deal as well with situations in which the optimizing action is not unique — in which the behavioral function is actually a correspondence. The Maximum Theorem is used pervasively in economics and game theory to infer that a behavioral correspondence is UHC or has a closed graph.

In applications of the Maximum Theorem, the set X in the statement of the theorem below is typically the action space; E is the set of possible environments (the parameter space); the function u is the objective function; the correspondence φ describes how the set of available actions depends upon the environment; and μ is the behavioral correspondence, describing how the individual's actions depend upon the environment he faces.

In demand theory, for example, X would be the consumption set (or a compact subset of it); E would be the set of possible price-lists (and perhaps wealth/income levels); u would be the consumer's utility function; φ would be the correspondence that determines the budget set from the market prices; and μ would be the consumer's demand correspondence.

The Maximum Theorem: Let X be a subset of $\mathbb{R}^l$; let E be a subset of $\mathbb{R}^m$; let $u : X \times E \to \mathbb{R}$ be a continuous function; and let $\varphi : E \twoheadrightarrow X$ be a continuous and compact-valued correspondence. Then the correspondence $\mu : E \twoheadrightarrow X$ defined by

$$\mu(e) = \{\, x \in \varphi(e) \,|\, x \text{ maximizes } u(\cdot, e) \text{ on } \varphi(e) \,\}$$

is nonempty-valued, compact-valued, closed, and UHC, and the value function $v : E \to \mathbb{R}$ defined by $v(e) = \max_{x \in \varphi(e)} u(x, e)$ is continuous.

Proof: Border, p. 64, or de la Fuente, p. 301.

Note that the domain of the objective function u includes not only actions (in the demand theory application these are consumption bundles), but parameters as well (in the demand application these are price-lists). This seems odd at first glance. There are three observations to make about this:

(1) In the demand theory application we typically assume that u is constant with respect to the parameters $e \in E$ — the prices. So for this application a Maximum Theorem in which u depends only upon consumption bundles in X would be just fine.

(2) Including the prices as arguments of u does, however, allow us to analyze consumers whose utility depends upon prices as well as consumption levels.

(3) The theory of the firm is an example of an application where we need to have the objective function u depend upon the parameters. In this application, the elements of X are the firm's feasible production plans, *i.e.*, input-output combinations. The elements of E are again price-lists. The objective function u is the firm's profit function — and note that profit does depend upon both the firm's choice of production plan (in X) and the market prices (in E). The correspondence φ describes how the available plans depend upon prices — typically, this is assumed to be a constant correspondence in the theory of the firm: the production plans available to the firm depend upon its technological capabilities but not upon prices. And of course the correspondence μ describes how the firm's profit-maximizing choices of input and output levels depend upon the market prices — the firm's supply correspondence for outputs and its demand correspondence for inputs.