THE MAXIMUM THEOREM

THIS THEOREM TELLS US HOW A DECISION-MAKER'S CHOICES (BEHAVIOR) RESPOND TO CHANGES IN THE DECISION-MAKING ENVIRONMENT.

THEOREM: LET

	IN APPLICATIONS
$X \subseteq \mathbb{R}^\ell$	$x \in X$ ARE ACTIONS
$E \subseteq \mathbb{R}^m$	$e \in E$ ARE ENVIRONMENTS/PARAMETERS
$u: X \times E \to \mathbb{R}$	OBJECTIVE FUNCTION
$\varphi: E \twoheadrightarrow X,$	FEASIBLE-SET CORRESPONDENCE

AND DEFINE

$$\mu: E \twoheadrightarrow X \text{ BY } \mu(e) = \{\hat{x} \in \varphi(e) \mid \hat{x} \text{ max's } u(\cdot, e) \text{ in } \varphi(e)\}$$

$$= \underset{x \in \varphi(e)}{\arg\max}\; u(\cdot, e)$$

BEHAVIORAL (SOLUTION) CORRESPONDENCE; STRATEGY

AND $V: E \to \mathbb{R}$ BY $V(e) = \max\limits_{x \in \varphi(e)} u(x, e)$. VALUE FUNCTION

IF u IS CONTINUOUS, AND

φ IS CONTINUOUS AND COMPACT-VALUED,

THEN

μ IS UHC AND NONEMPTY-VALUED, AND

V IS CONTINUOUS.

REMARK: SINCE μ IS CLOSED-VALUED AND UHC, μ HAS A CLOSED GRAPH.

REMARK: IF μ IS SINGLETON-VALUED (A FUNCTION), IT IS CONTINUOUS.

THE MAXIMUM THEOREM IN DEMAND THEORY

MAXIMUM THEOREM	DEMAND THEORY
$x \in X \subseteq \mathbb{R}^\ell$	$x \in \mathbb{R}^\ell_+$, CONSUMPTION BUNDLES
$e \in E \subseteq \mathbb{R}^m$	$p \in \mathbb{R}^\ell_+$, PRICE LISTS
$u: \cancel{X} X \times E \to \mathbb{R}$	$u: \mathbb{R}^\ell_+ \times \mathbb{R}^\ell_+ \to \mathbb{R}$, UTILITY FUNCTION ← CONSTANT W.R.T. 2ND COMPONENT
$\varphi: E \twoheadrightarrow X$	$B(p,m) = \{x \in \mathbb{R}^\ell_+ \mid p \cdot x \leqq m\}$, m COULD BE $p \cdot \mathring{x}$, CONSUMER'S BUDGET SET
$\mu: E \twoheadrightarrow X$	$\mu: \mathbb{R}^\ell_+ \twoheadrightarrow \mathbb{R}^\ell_+$, CONSUMER'S DEMAND CORRESPONDENCE
$v: E \to \mathbb{R}$	$v: \mathbb{R}^\ell_+ \to \mathbb{R}$, INDIRECT UTILITY FUNCTION

THE MAXIMUM THEOREM TELLS US THAT THE CONSUMER'S DEMAND CORRESPONDENCE $\mu: \mathbb{R}^\ell_+ \twoheadrightarrow \mathbb{R}^\ell_+$ IS NONEMPTY-VALUED, COMPACT-VALUED, CLOSED, AND UHC IF THE BUDGET CORRESPONDENCE IS CONTINUOUS AND COMPACT-VALUED. SINCE THE BUDGET CORRESPONDENCE IS TYPICALLY NOT COMPACT-VALUED FOR PRICE-LISTS THAT HAVE SOME $p_k = 0$, WE NEED TO EITHER RESTRICT PRICES TO BE IN A SUBSET $\mathcal{P}$ OF $\mathbb{R}^\ell_{++}$, OR ELSE RESTRICT THE CONSUMPTION SET X TO A COMPACT SUBSET OF $\mathbb{R}^\ell_+$.

WE ALSO OBTAIN:

THE INDIRECT UTILITY FUNCTION IS CONTINUOUS, AND IF DEMAND $\mu(p)$ IS SINGLETON-VALUED, THEN THE DEMAND FUNCTION IS CONTINUOUS.

THEORY OF THE FIRM:

A FIRM'S TECHNOLOGY (ITS TECHNOLOGICALLY FEASIBLE PRODUCTION PLANS) IS REPRESENTED BY ITS PRODUCTION SET $T \subseteq \mathbb{R}^\ell$:

$x_k > 0$: GOOD k IS AN OUTPUT AT x

$x_k < 0$: GOOD k IS AN INPUT AT x

T IS THE SET OF ALL TECHNOLOGICALLY FEASIBLE PRODUCTION PLANS

PROFIT AT PLAN x (AT PRICE-LIST $p \in \mathbb{R}^\ell_+$):

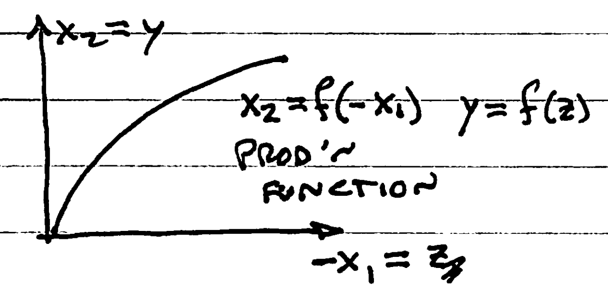
$$\pi(x;p) := p \cdot x$$

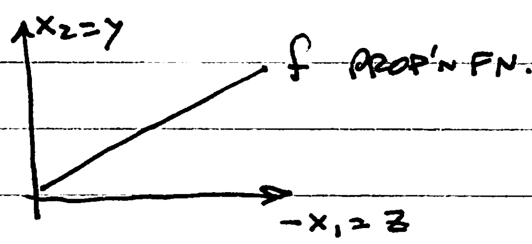
$$= \sum_{k \in O} p \cdot x_k + \sum_{k \in I} p \cdot x_k \qquad - \sum_{k \in I} p \cdot \underbrace{(-x_k)}_{>0}.$$

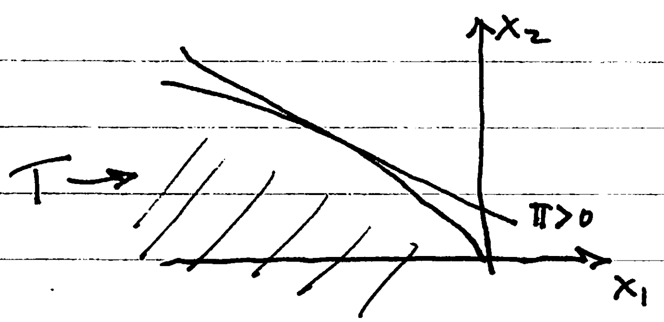
$$O = \{k \mid x_k \geq 0\}, \quad I = \{k \mid x_k \leq 0\}$$

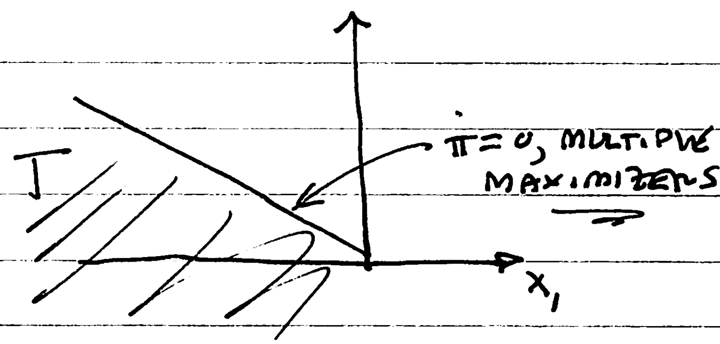
$$= \text{REVENUE} - \text{COST}.$$

CONSTANT RETURNS TO SCALE:









The Firm's Behavioral Correspondence

(Input-Output Correspondence)

Firm's Demand ↗ ↖ Firm's Supply

Maximum Theorem	Theory of the Firm
$x \in X \subseteq \mathbb{R}^\ell$	$x \in X \subseteq \mathbb{R}^\ell$, input-output plans ← in the production set X
$e \in E \subseteq \mathbb{R}^m$	$p \in \mathbb{R}^\ell_+$, price-lists
$u: X \times E \to \mathbb{R}$	$\pi(x,p) = p \cdot x$, Firm's profit
$\varphi: E \twoheadrightarrow X$	$\varphi(p) = X$, a constant correspondence; production set isn't affected by prices.
$\mu: E \twoheadrightarrow X$	$\mu(p) = \{x \in X \mid x \text{ max's } \pi \text{ on } X\}$
$v: E \to \mathbb{R}$	$v(p) = \max_{x \in X} \pi(x,p) = \max_{x \in X} p \cdot x$, – Firm's profit as function of p.

The Maximum Theorem tells us that the Firm's supply/demand correspondence μ is nonempty-valued, compact-valued, uhc, and closed IF the production set X is compact. Since it typically isn't, we typically need to restrict it to a compact set when applying the Maximum Theorem.

THE MAXIMUM THEOREM

IN OUR GROWTH THEORY EXAMPLE

RECALL THAT WE WANTED TO ESTABLISH THE EXISTENCE OF A FUNCTION $v : \mathbb{R}_+ \to \mathbb{R}_+$ THAT SATISFIES

CONT'S CONT'S

$$(*) \quad \forall x \in \mathbb{R}_+ : v(x) = \max_{z \in \mathbb{R}_+} \left[u(f(x) - z) + \beta v(z) \right] \text{ s.t. } 0 \leqq z \leqq f(x).$$

NOTE THAT v — THE <u>SAME</u> FUNCTION v — IS ON BOTH SIDES OF THIS (FUNCTIONAL) EQUATION. FOR ANY $x \in \mathbb{R}_+$, $v(x)$ WAS THE VALUE, ~~[illegible]~~ AT AN ARBITRARY TIME t, OF HAVING CAPITAL STOCK x — THE <u>PRESENT</u> VALUE AT t OF THE CURRENT AND FUTURE STREAM OF PERIOD-BY-PERIOD VALUES.

IN ORDER TO SHOW THAT THERE <u>IS</u> SUCH A v, WE DESCRIBED $v(\cdot)$ AS THE FIXED POINT OF A TRANSFORMATION $T : \mathcal{F} \to \mathcal{F}$ THAT MAPS FUNCTIONS $v \in \mathcal{F}$ ($\mathcal{F}$ IS A SET OF FUNCTIONS $v : \mathbb{R}_+ \to \mathbb{R}_+$) INTO OTHER FUNCTIONS, SAY $\tilde{v} \in \mathcal{F}$. WE DEFINED THE TRANSFORMATION T (OR EQUIVALENTLY, THE FUNCTION $\tilde{v}$ FOR A GIVEN v) AS FOLLOWS:

$\tilde{v} = T(v)$

$$(\tilde{*}) \quad \tilde{v}(x) = \max_{z \in \mathbb{R}_+} \left[u(f(x) - z) + \beta v(z) \right] \text{ s.t. } 0 \leqq z \leqq f(x).$$

CLEARLY, A FUNCTION $v : \mathbb{R}_+ \to \mathbb{R}_+$ SATISFIES $(*)$ IF AND ONLY IF IT'S A FIXED POINT OF T — IF AND ONLY IF $\tilde{v} = v$.

WE WANT TO ESTABLISH THAT $T: \mathcal{F} \to \mathcal{F}$ HAS A FIXED POINT, WHERE $\mathcal{F}$ IS THE SET OF ALL FUNCTIONS $v: \mathbb{R}_+ \to \mathbb{R}$ — WE WANT TO SHOW THERE IS SOME FUNCTION $v: \mathbb{R}_+ \to \mathbb{R}$ FOR WHICH $\tilde{v} = v$. WE'LL USE A FIXED POINT THEOREM TO ESTABLISH THIS, BUT WE DON'T HAVE A THEOREM THAT APPLIES TO TO THE SET $\mathcal{F}$ OF ALL FUNCTIONS FROM $\mathbb{R}_+$ INTO $\mathbb{R}$. SO WE NEED TO NARROW DOWN THE SET $\mathcal{F}$. IF WE CAN NARROW IT DOWN TO A SET $\mathcal{F}' \subseteq \mathcal{F}$ FOR WHICH A FIXED POINT THEOREM DOES APPLY, WE'LL HAVE ESTABLISHED THAT $\mathcal{F}'$ (AND THEREFORE $\mathcal{F}$) DOES HAVE A FIXED POINT.

WHAT SET $\mathcal{F}'$ SHOULD WE USE? IF WE WANT TO USE THE BANACH FIXED POINT THEOREM, FOR EXAMPLE, $\mathcal{F}'$ WILL HAVE TO SATISFY THE FOLLOWING CONDITIONS:

(1) $\mathcal{F}'$ IS A COMPLETE METRIC SPACE (IN SOME METRIC)

(2) IF $v \in \mathcal{F}'$ THEN $\tilde{v} \in \mathcal{F}'$ — i.e., T MAPS $\mathcal{F}'$ INTO $\mathcal{F}'$.

(3) T IS A CONTRACTION ON $\mathcal{F}'$ — i.e., IN THE METRIC FOR (1), $\exists \beta < 1$ SUCH THAT

$$\forall v, w \in \mathcal{F}': d(\tilde{v}, \tilde{w}) \leqq \beta d(v, w).$$

WE USE $C_B(\mathbb{R}_+)$, THE SET OF ALL BOUNDED CONTINUOUS FUNCTIONS ON $\mathbb{R}_+$ AND THE sup-METRIC ON $C_B(\mathbb{R}_+)$:

(1) IS EASY TO SHOW.

(3) IS RELATIVELY STRAIGHTFORWARD. (WE WON'T DO IT HERE.)

(2a) $\tilde{v}$ IS BOUNDED IS EASY TO SHOW.

(2b) $\tilde{v}$ IS CONTINUOUS: WE'LL DO THIS HERE.

WE SHOW THAT

(**) $v \in C_B(\mathbb{R}_+) \Rightarrow \tilde{v}$ IS CONTINUOUS.

NOTE THAT THIS IS ANOTHER APPLICATION IN WHICH WE NEED $X \times E$ TO BE THE DOMAIN OF u

MAXIMUM THEOREM	GROWTH THEORY APPLICATION
$x \in X \subseteq \mathbb{R}^\ell$	$z \in \mathbb{R}_+$
$e \in E \subseteq \mathbb{R}^m$	$x \in \mathbb{R}_+$
$u: X \times E \to \mathbb{R}$	$\tilde{u}(z,x) = u(f(x)-z) + \beta v(z)$
$\varphi: E \twoheadrightarrow X$	$\varphi(x) = [0, f(x)]$
$\mu: E \twoheadrightarrow X$	$\mu(x) = \{z \in \mathbb{R}_+ \mid z$ IS A SOLUTION OF THE MAX-PROBLEM IN (*)$\}$
$v: E \to \mathbb{R}$	$\tilde{v}(x)$. IN (*)

$\mu(x) = \underset{z \in \varphi(x)}{\arg\max}\ \tilde{u}(z,x)$

(WE ASSUME $v \in C(\mathbb{R}_+)$)

$\tilde{u}$ IS CLEARLY CONTINUOUS IF f AND u ARE CONTINUOUS.

EXERCISE → EASY TO SHOW THAT φ IS CONTINUOUS (UHC & LHC).

φ IS OBVIOUSLY COMPACT-VALUED.

$\therefore$ μ IS UHC AND $\underbrace{\tilde{v} \text{ IS CONTINUOUS}}_{\text{THIS IS WHAT WE NEED: (**) IS SATISFIED}}$.

THEREFORE THE CONTRACTION MAPPING THEOREM APPLIES, AND THERE IS A FIXED POINT OF T — A $v: \mathbb{R}_+ \to \mathbb{R}_+$ THAT SATISFIES (*).