2 *An Overview*

In this chapter we preview the recursive methods of analysis to be developed in detail in the rest of the book. This material falls into three broad parts, and the remainder of the book is structured accordingly. Part II deals with methods for solving deterministic optimization problems, Part III with the extension of these methods to problems that include stochastic shocks, and Part IV with ways of using solutions of either type within a competitive equilibrium framework.

To make this preview as concrete as possible, we examine these three sets of issues by looking at a specific example, a one-sector model of economic growth. Our goal is not to provide a substantive treatment of growth theory but to illustrate the types of arguments and results that are developed in the later chapters of the book—arguments that can be applied to a wide variety of problems. A few of these problems were mentioned in Chapter 1, and many more will be discussed in detail in Chapters 5, 10, 13, 16, 17, and 18, all of which are devoted exclusively to substantive applications. With that said, in this chapter we focus exclusively on the example of economic growth.

In the next three sections we consider resource allocation in an economy composed of many identical, infinitely lived households. In each period t there is a single good, y_t, that is produced using two inputs: capital, k_t, in place at the beginning of the period, and labor, n_t. A production function relates output to inputs, $y_t = F(k_t, n_t)$. In each period current output must be divided between current consumption, c_t, and gross investment, i_t,

(1) $$c_t + i_t \leq y_t = F(k_t, n_t).$$

This consumption-savings decision is the only allocation decision the economy must make. Capital is assumed to depreciate at a constant rate

$0 < \delta < 1$, so capital is related to gross investment by

(2) $$k_{t+1} = (1 - \delta)k_t + i_t.$$

Labor is taken to be supplied inelastically, so $n_t = 1$, all t. Finally, prefer ences over consumption, common to all households, are taken to be o the form

(3) $$\sum_{t=0}^{\infty} \beta^t U(c_t),$$

where $0 < \beta < 1$ is a discount factor.

In Sections 2.1 and 2.2 we study the problem of optimal growth Specifically, in Section 2.1 we examine the problem of maximizing (3 subject to (1) and (2), given an initial capital stock k_0. In Section 2.2 w modify this planning problem to include exogenous random shocks t the technology in (1), in this case taking the preferences of household over random consumption sequences to be the expected value of th function in (3). In Section 2.3 we return to the deterministic model. W begin by characterizing the paths for consumption and capital accumula tion that would arise in a competitive market economy composed o many households, each with the preferences in (3), and many firms, eac with the technology in (1) and (2). We then consider the relationshi between the competitive equilibrium allocation and the solution to th planning problem found earlier. We conclude in Section 2.4 with a mor detailed overview of the remainder of the book, discussing briefly th content of each of the later chapters.

2.1 A Deterministic Model of Optimal Growth

In this section we study the problem of optimal growth when there is n uncertainty. Assume that the production function is $y_t = F(k_t, n_t)$, wher $F: \mathbf{R}^2_+ \to \mathbf{R}_+$ is continuously differentiable, strictly increasing, homoge neous of degree one, and strictly quasi-concave, with

$$F(0, n) = 0, \quad F_k(k, n) > 0, \quad F_n(k, n) > 0, \quad \text{all } k, n > 0;$$

$$\lim_{k \to 0} F_k(k, 1) = \infty, \quad \lim_{k \to \infty} F_k(k, 1) = 0.$$

Assume that the size of the population is constant over time and normalize the size of the available labor force to unity. Then actual labor supply must satisfy

(1a) $$0 \leq n_t \leq 1, \quad \text{all } t.$$

Assume that capital decays at the fixed rate $0 < \delta \leq 1$. Then consumption c_t, gross investment $i_t = k_{t+1} - (1 - \delta)k_t$, and output $y_t = F(k_t, n_t)$ must satisfy the feasibility constraint

(1b) $$c_t + k_{t+1} - (1 - \delta)k_t \leq F(k_t, n_t), \quad \text{all } t.$$

Assume that all of the households in this economy have identical preferences over intertemporal consumption sequences. These common preferences take the additively separable form

(2) $$u(c_0, c_1, \ldots) = \sum_{t=0}^{\infty} \beta^t U(c_t),$$

where the discount factor is $0 < \beta < 1$, and where the current-period utility function $U: \mathbf{R}_+ \to \mathbf{R}$ is bounded, continuously differentiable, strictly increasing, and strictly concave, with $\lim_{c\to 0} U'(c) = \infty$. Households do not value leisure.

Now consider the problem faced by a benevolent social planner, one whose objective is to maximize (2) by choosing sequences $\{(c_t, k_{t+1}, n_t)\}_{t=0}^{\infty}$, subject to the feasibility constraints in (1), given $k_0 > 0$. Two features of any optimum are apparent. First, it is clear that output will not be wasted. That is, (1b) will hold with equality for all t, and we can use it to eliminate c_t from (2). Second, since leisure is not valued and the marginal product of labor is always positive, it is clear that an optimum requires $n_t = 1$, all t. Hence k_t and y_t represent both capital and output per worker and capital and output in total. It is therefore convenient to define $f(k) = F(k, 1) + (1 - \delta)k$ to be the total supply of goods available per worker, including undepreciated capital, when beginning-of-period capital is k.

Exercise 2.1 Show that the assumptions on F above imply that $f: \mathbf{R}_+ \to \mathbf{R}_+$ is continuously differentiable, strictly increasing, and strictly

concave, with

$$f(0) = 0, \quad f'(k) > 0, \quad \lim_{k\to 0} f'(k) = \infty, \quad \lim_{k\to\infty} f'(k) = 1 - \delta.$$

The planning problem can then be written as

(3) $$\max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U[f(k_t) - k_{t+1}]$$

(4) $$\text{s.t.} \quad 0 \leq k_{t+1} \leq f(k_t), \quad t = 0, \ldots;$$

$$k_0 > 0 \quad \text{given.}$$

Although ultimately we are interested in the case where the planning horizon is infinite, it is instructive to begin with the (much easier!) problem of a finite horizon. If the horizon in (3) were a finite value T instead of infinity, then (3)–(4) would be an entirely standard concave programming problem. With a finite horizon, the set of sequences $\{k_{t+1}\}_{t=0}^{T}$ satisfying (4) is a closed, bounded, and convex subset of $\mathbf{R}^{T+1}$, and the objective function (3) is continuous and strictly concave. Hence there is exactly one solution, and it is completely characterized by the Kuhn-Tucker conditions.

To obtain these conditions note that since $f(0) = 0$ and $U'(0) = \infty$, it is clear that the inequality constraints in (4) do not bind except for k_{T+1}, and it is also clear that $k_{T+1} = 0$. Hence the solution satisfies the first-order and boundary conditions

(5) $$\beta f'(k_t)U'[f(k_t) - k_{t+1}] = U'[f(k_{t-1}) - k_t], \quad t = 1, 2, \ldots, T;$$

(6) $$k_{T+1} = 0, \quad k_0 > 0 \quad \text{given.}$$

Equation (5) is a second-order difference equation in k_t; hence it has a two-parameter family of solutions. The unique optimum for the maximization problem of interest is the one solution in this family that in addition satisfies the two boundary conditions in (6). The following exercise illustrates how (5)–(6) can be used to solve for the optimum in a particular example.

Exercise 2.2 Let $f(k) = k^\alpha$, $0 < \alpha < 1$, and let $U(c) = \ln(c)$. (No, this does not fit all of the assumptions we placed on f and U above, but go ahead anyway.)

a. Write (5) for this case and use the change of variable $z_t = k_t/k_{t-1}^\alpha$ to convert the result into a first-order difference equation in z_t. Plot z_{t+1} against z_t and plot the 45° line on the same diagram.

b. The boundary condition (6) implies that $z_{T+1} = 0$. Using this condition, show that the unique solution is

$$z_t = \alpha\beta \frac{1 - (\alpha\beta)^{T-t+1}}{1 - (\alpha\beta)^{T-t+2}}, \quad t = 1, 2, \ldots, T + 1.$$

c. Check that the path for capital

$$k_{t+1} = \alpha\beta \frac{1 - (\alpha\beta)^{T-t}}{1 - (\alpha\beta)^{T-t+1}} k_t^\alpha, \quad t = 0, 1, \ldots, T, \tag{7}$$

given k_0, satisfies (5)–(6).

Now consider the infinite-horizon version of the planning problem in Exercise 2.2. Note that if T is large, then the coefficient of k_t^α in (7) is essentially constant at $\alpha\beta$ for a very long time. For the solution to the infinite-horizon problem, can we not simply take the limit of the solutions in (7) as T approaches infinity? After all, we are discussing households that discount the future at a geometric rate! Taking the limit in (7), we find that

$$k_{t+1} = \alpha\beta k_t^\alpha, \quad t = 0, 1, \ldots. \tag{8}$$

In fact, this conjecture is correct: the limit of the solutions for the finite-horizon problems is the unique solution to the infinite-horizon problem. This is true both for the parametric example in Exercise 2.2 and for the more generally posed problem. But proving it involves establishing the legitimacy of interchanging the operators "max" and "$\lim_{T\to\infty}$"; and doing this is more challenging than one might guess.

Instead we will pursue a different approach. Equation (8) suggests another conjecture: that for the infinite-horizon problem in (3)–(4), for any U and f, the solution takes the form

$$k_{t+1} = g(k_t), \quad t = 0, 1, \ldots, \tag{9}$$

where g: $\mathbf{R}_+ \to \mathbf{R}$ is a fixed savings function. Our intuition suggests that this must be so: since the planning problem takes the same form every period, with only the beginning-of-period capital stock changing from one period to the next, what else but k_t could influence the choice of k_{t+1} and c_t? Unfortunately, Exercise 2.2 does not offer any help in pursuing this conjecture. The change of variable exploited there is obviously specific to the particular functional forms assumed, and a glance at (5) confirms that no similar method is generally applicable.

The strategy we *will* use to pursue this idea involves ignoring (5) and (6) altogether and starting afresh. Although we stated this problem as one of choosing infinite sequences $\{(c_t, k_{t+1})\}_{t=0}^\infty$ for consumption and capital, the problem that in fact faces the planner in period $t = 0$ is that of choosing today's consumption, c_0, and tomorrow's beginning-of-period capital, k_1, and nothing else. The rest can wait until tomorrow. If we knew the planner's preferences over these two goods, we could simply maximize the appropriate function of (c_0, k_1) over the opportunity set defined by (1b), given k_0. But what are the planner's preferences over current consumption and next period's capital?

Suppose that (3)–(4) had already been solved for all possible values of k_0. Then we could define a function v: $\mathbf{R}_+ \to \mathbf{R}$ by taking $v(k_0)$ to be the value of the maximized objective function (3), for each $k_0 \geq 0$. A function of this sort is called a *value function.* With v so defined, $v(k_1)$ would give the value of the utility from period 1 on that could be obtained with a beginning-of-period capital stock k_1, and $\beta v(k_1)$ would be the value of this utility discounted back to period 0. Then in terms of this value function v, the planner's problem in period 0 would be

$$\max_{c_0, k_1} [U(c_0) + \beta v(k_1)] \tag{10}$$

$$\text{s.t.} \quad c_0 + k_1 \leq f(k_0),$$

$$c_0, k_1 \geq 0, \quad k_0 > 0 \quad \text{given.}$$

If the function v were known, we could use (10) to define a function g: $\mathbf{R}_+ \to \mathbf{R}_+$ as follows: for each $k_0 \geq 0$, let $k_1 = g(k_0)$ and $c_0 = f(k_0) - g(k_0)$ be the values that attain the maximum in (10). With g so defined, (9) would completely describe the dynamics of capital accumulation from any given initial stock k_0.

We do not at this point "know" v, but we have defined it as the maximized objective function for the problem in (3)–(4). Thus, if solving (10) provides the solution for that problem, then $v(k_0)$ must be the maximized objective function for (10) as well. That is, v must satisfy

$$v(k_0) = \max_{0 \le k_1 \le f(k_0)} \{U[f(k_0) - k_1] + \beta v(k_1)\},$$

where, as before, we have used the fact that goods will not be wasted.

Notice that when the problem is looked at in this recursive way, the time subscripts have become a nuisance: we do not care what the date is. We can rewrite the problem facing a planner with current capital stock k as

$$(11) \qquad v(k) = \max_{0 \le y \le f(k)} \{U[f(k) - y] + \beta v(y)\}.$$

This one equation in the unknown function v is called a *functional equation,* and we will see later that it is a very tractable mathematical object. The study of dynamic optimization problems through the analysis of such functional equations is called *dynamic programming.*

If we knew that the function v was differentiable and that the maximizing value of y—call it $g(k)$—was interior, then the first-order and envelope conditions for (11) would be

$$U'[f(k) - g(k)] = \beta v'[g(k)], \quad \text{and}$$

$$v'(k) = f'(k)U'[f(k) - g(k)],$$

respectively. The first of these conditions equates the marginal utility of consuming current output to the marginal utility of allocating it to capital and enjoying augmented consumption next period. The second condition states that the marginal value of current capital, in terms of total discounted utility, is given by the marginal utility of using the capital in current production and allocating its return to current consumption.

Exercise 2.3 We conjectured that the path for capital given by (8) was optimal for the infinite-horizon planning problem, for the functional forms of Exercise 2.2.

a. Use this conjecture to calculate v by evaluating (2) along the consumption path associated with the path for capital given by (8).

b. Verify that this function v satisfies (11).

c. Is this a proof that (8) gives the optimal policy for this case? What would be needed to make it into one?

Suppose we have established the existence of an optimal savings policy g, either by analyzing conditions (5)–(6) or by analyzing the functional equation (11). What can we do with this information? For the particular parametric example in Exercises 2.2 and 2.3, we can solve for g with pencil-and-paper methods. We can then use the resulting difference equation (8) to compute the optimal sequence of capital stocks $\{k_t\}$. This example is a carefully chosen exception: for most other parametric examples, it is not possible to obtain an explicit analytical solution for the savings function g. In such cases a numerical approach can be used to compute explicit solutions. When all parameters are specified numerically, it is possible to use an algorithm based on (11) to obtain an approximation to g. Then $\{k_t\}$ can be computed using (9), given any initial value k_0.

In addition, there are often qualitative features of the savings function g, and hence of the capital paths generated by (9), that hold under a very wide range of assumptions on f and U. Specifically, we can use either (5)–(6) or the first-order and envelope conditions for (11), together with assumptions on U and f, to characterize the optimal savings function g. We can then, in turn, use the properties of g so established to characterize solutions $\{k_t\}$ to (9). The following exercise illustrates the second of these steps.

Exercise 2.4 a. Let f be as specified in Exercise 2.1, and suppose that the optimal savings function g is characterized by a constant savings rate, $g(k) = sf(k)$, all k, where $s > 0$. Plot g, and on the same diagram plot the 45° line. The points at which $g(k) = k$ are called the *stationary solutions, steady states, rest points,* or *fixed points* of g. Prove that there is exactly one positive stationary point k^*.

b. Use the diagram to show that if $k_0 > 0$, then the sequence $\{k_t\}$ given by (9) converges to k^* as $t \to \infty$. That is, let $\{k_t\}_{t=0}^{\infty}$ be a sequence satisfying (9), given some $k_0 \ge 0$. Prove that $\lim_{t\to\infty} k_t = k^*$, for any $k_0 > 0$. Show that this convergence is monotonic. Can it occur in a finite number of periods?

This exercise contains most of the information that can be established about the qualitative behavior of a sequence generated by a deterministic dynamic model. The stationary points have been located and characterized, their stability properties established, and the motion of the system has been described qualitatively for all possible initial positions. We take this example as a kind of image of what one might hope to establish for more complicated models, or as a source of reasonable conjectures. (Information about the rate of convergence to the steady state k^*, for k_t near k^*, can be obtained by taking a linear approximation to g in a neighborhood of k^*. Alternatively, numerical simulations can be used to study the rate of convergence over any range of interest.)

From the discussion above, we conclude that a fruitful way of analyzing a stationary, infinite-horizon optimization problem like the one in (3)–(4) is by examining the associated functional equation (11) for this example—and the difference equation (9) involving the associated policy function. Several steps are involved in carrying out this analysis.

First we need to be sure that the solution(s) to a problem posed in terms of infinite sequences are also the solution(s) to the related functional equation. That is, we need to show that by using the functional equation we have not changed the problem. Then we must develop tools for studying equations like (11). We must establish the existence and uniqueness of a value function v satisfying the functional equation and, where possible, to develop qualitative properties of v. We also need to establish properties of the associated policy function g. Finally we must show how qualitative properties of g are translated into properties of the sequences generated by g.

Since a wide variety of problems from very different substantive areas of economics all have this same mathematical structure, we want to develop these results in a way that is widely applicable. Doing this is the task of Part II.

2.2 A Stochastic Model of Optimal Growth

The deterministic model of optimal growth discussed above has a variety of stochastic counterparts, corresponding to different assumptions about the nature of the uncertainty. In this section we consider a model in which the uncertainty affects the technology only, and does so in a specific way.

Assume that output is given by $y_t = z_t f(k_t)$ where $\{z_t\}$ is a sequence of

independently and identically distributed (i.i.d.) random variables, and f is defined as it was in the last section. The shocks may be thought of as arising from crop failures, technological breakthroughs, and so on. The feasibility constraints for the economy are then

$$(1) \qquad k_{t+1} + c_t \leq z_t f(k_t), \quad c_t, k_{t+1} \geq 0, \quad \text{all } t, \text{ all } \{z_t\}.$$

Assume that the households in this economy rank stochastic consumption sequences according to the expected utility they deliver, where their underlying (common) utility function takes the same additively separable form as before:

$$(2) \qquad \mathrm{E}[u(c_0, c_1, \ldots)] = \mathrm{E}\left[\sum_{t=0}^{\infty} \beta^t U(c_t)\right].$$

Here $\mathrm{E}(\cdot)$ denotes expected value with respect to the probability distribution of the random variables $\{c_t\}_{t=0}^{\infty}$.

Now consider the problem facing a benevolent social planner in this stochastic environment. As before, his objective is to maximize the objective function in (2) subject to the constraints in (1). Before proceeding, we need to be clear about the timing of information, actions, and decisions, about the objects of choice for the planner, and about the distribution of the random variables $\{c_t\}_{t=0}^{\infty}$.

Assume that the timing of information and actions in each period is as follows. At the beginning of period t the current value z_t of the exogenous shock is realized. Thus, the pair (k_t, z_t), and hence the value of total output $z_t f(k_t)$, are known when consumption c_t takes place and end-of-period capital k_{t+1} is accumulated. The pair (k_t, z_t) is called the *state* of the economy at date t.

As we did in the deterministic case, we can think of the planner in period 0 as choosing, in addition to the pair (c_0, k_1), an infinite sequence $\{(c_t, k_{t+1})\}_{t=1}^{\infty}$ describing all future consumption and capital pairs. In the stochastic case, however, this is not a sequence of numbers but a sequence of *contingency plans*, one for each period. Specifically, consumption c_t and end-of-period capital k_{t+1} in each period $t = 1, 2, \ldots$ are contingent on the realizations of the shocks $z_1, z_2, \ldots, z_t$. This sequence of realizations is information that is available when the decision is being carried out but is unknown in period 0 when the decision is being made.